Tangent Lines and the Derivative at a Point

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Outline

Tangent Lines and the Derivative at a Point

- As usual, you should read section 3.1 in the online textbook.
- This slideshow will give an overview and an explanation of the important concepts in the book.
- This slideshow will also include a limited number of examples.
- The main purpose of this slideshow is to give an extended explanation and clarification of the material in the text.

Finding a Tangent Line to the Graph of a Function

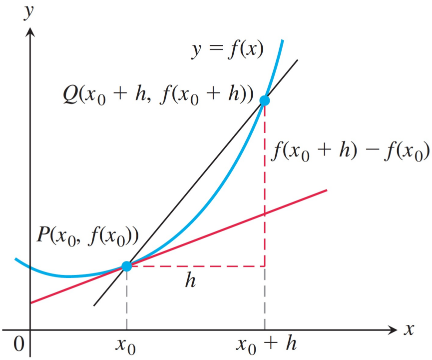

Figure: Slope of the tangent line at P

To find the slope of the tangent line to an arbitrary curve $y = f(x)$ at a point $P(x_0, f(x_0))$, we calculate the slope of the secant line through P and a nearby point $Q(x_0 + h, f(x_0 + h))$. We then investigate the limit of the slope as $h \to 0$.

Finding a Tangent Line to the Graph of a Function

Definition

The **slope of the curve** $y = f(x)$ at the point $P(x_0, f(x_0))$ is the number

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h},$$

provided this limit exists.

The **tangent line** to the curve at P is the line through P with this slope.

Example 1

Example

Find an equation for the tangent line to the curve $y = 4 - x^2$ at the point $(-1, 3)$. Then sketch the curve and tangent line together.

Solution

We compute the slope of the tangent line:

$$\begin{aligned}
\lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{h} &= \lim_{h\to 0} \frac{f(-1+h) - f(-1)}{h} \\
&= \lim_{h\to 0} \frac{[4 - (-1+h)^2] - 3}{h} \\
&= \lim_{h\to 0} \frac{[4 - (1 - 2h + h^2)] - 3}{h} \\
&= \lim_{h\to 0} \frac{2h - h^2}{h} = \lim_{h\to 0} \frac{h(2-h)}{h} \\
&= \lim_{h\to 0} (2 - h) = 2.
\end{aligned}$$

Example 1

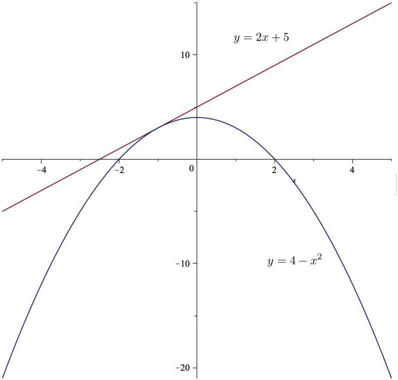

Figure: Sketch for Example 1

The equation of the tangent line is then

$$y - y_1 = m(x - x_1)$$
$$y - 3 = 2(x - (-1))$$
$$y = 2x + 5.$$

Rates of Change: Derivative at a Point

The expression

$$\frac{f(x_0+h)-f(x_0)}{h}, \quad h \neq 0$$

is the **difference quotient of** f **at** x_0 **with increment** h.

Definition

The **derivative of a function** f **at a point** x_0, denoted $f'(x_0)$, is

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0+h)-f(x_0)}{h},$$

provided this limit exists.

Rates of Change: Derivative at a Point

The derivative has another interpretation in certain physical applications.

If we interpret the difference quotient as the average rate of change of the function f on the interval $[x_0, x_0 + h]$, then the derivative $f'(x_0)$ is the **instantaneous rate of change of f at x_0.**

Example 2

Example

A rock falls $y = 16t^2$ feet during the first t seconds. What is the rock's exact speed after 1 second?

Solution

With $f(t) = 16t^2$*, the difference quotient*

$$\frac{f(1+h) - f(1)}{h}$$

is the average velocity over the interval $[1, 1+h]$.

Example 2

Solution

If we then let $h \to 0$, we get the rock's exact speed:

$$\begin{aligned}
\lim_{h\to 0} \frac{f(1+h)-f(1)}{h} &= \lim_{h\to 0} \frac{16(1+h)^2-16}{h} \\
&= \lim_{h\to 0} \frac{16(1+2h+h^2)-16}{h} \\
&= \lim_{h\to 0} \frac{16+32h+16h^2-16}{h} \\
&= \lim_{h\to 0} \frac{32h+16h^2}{h} \\
&= \lim_{h\to 0} \frac{h(32+16h)}{h} \\
&= \lim_{h\to 0} 32+16h \\
&= 32 \ ft/s.
\end{aligned}$$

Summary

Summary

The following are all interpretations for the limit of the difference quotient

$$\lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h}.$$

1. The slope of the graph of $y = f(x)$ at $x = x_0$.
2. The slope of the tangent line to the curve $y = f(x)$ at $x = x_0$.
3. The rate of change of $f(x)$ with respect to x at $x = x_0$.
4. The derivative $f'(x_0)$ at $x = x_0$.