Limits Involving Infinity; Asymptotes of Graphs

William M. Faucette

University of West Georgia

Outline

Limits Involving Infinity; Asymptotes of Graphs

- As usual, you should read section 2.6 in the online textbook.
- This slideshow will give an overview and an explanation of the important concepts in the book.
- This slideshow will also include a limited number of examples.
- The main purpose of this slideshow is to give an extended explanation and clarification of the material in the text.

Finite Limits as $x \to \pm\infty$

Definition

We say that $f(x)$ has the **limit** L **as** x **approaches infinity** and write

$$\lim_{x \to \infty} f(x) = L$$

if, for every number $\varepsilon > 0$, there exists a corresponding number M such that for all x in the domain of f

$$|f(x) - L| < \varepsilon \quad \text{whenever} \quad x > M.$$

This means that if x gets very, very large, $f(x)$ gets closer and closer to L.

Finite Limits as $x \to \pm\infty$

Definition

We say that $f(x)$ has the **limit** L **as** x **approaches minus infinity** and write

$$\lim_{x \to -\infty} f(x) = L$$

if, for every number $\varepsilon > 0$, there exists a corresponding number N such that for all x in the domain of f

$$|f(x) - L| < \varepsilon \quad \text{whenever} \quad x < N.$$

This means that if x gets very, very large and negative, $f(x)$ gets closer and closer to L.

Example 1

Example

The function $f(x) = \frac{1}{x}$ has the limit zero as x goes to infinity or to minus infinity.

$$\lim_{x\to\infty} \frac{1}{x} = 0$$
$$\lim_{x\to-\infty} \frac{1}{x} = 0.$$

Properties of Limits

All the Limit Laws in Section 2.2 remain true if $x \to c$ is replaced by $x \to \infty$ and $x \to -\infty$. For example, we have the following:

Theorem (The Limit Laws)

If $\lim_{x\to\infty} f(x) = L$ *and* $\lim_{x\to\infty} g(x) = M$ *for numbers* k, L, *and* M, *then*

Sum Rule:	$\lim\limits_{x\to\infty} (f(x) + g(x)) = L + M$
Difference Rule:	$\lim\limits_{x\to\infty} (f(x) - g(x)) = L - M$
Constant Multiple Rule:	$\lim\limits_{x\to\infty} (kf(x)) = kL$
Product Rule:	$\lim\limits_{x\to\infty} (f(x)g(x)) = LM$
Quotient Rule:	$\lim\limits_{x\to\infty} \dfrac{f(x)}{g(x)} = \dfrac{L}{M}$, *if* $M \neq 0$
Power Rule:	$\lim\limits_{x\to\infty} [f(x)]^n = L^n$
Root Rule:	$\lim\limits_{x\to\infty} \sqrt[n]{f(x)} = \sqrt[n]{L}$, *if* n *is a positive integer*

(In the last rule if n *is even,* $f(x)$ *must be nonnegative near* c*.)*

Example 2

Example

1. $$\lim_{x\to\infty}\left(8-\frac{1}{x^2}\right)=\lim_{x\to\infty}8-\lim_{x\to\infty}\frac{1}{x^2}=\lim_{x\to\infty}8-\lim_{x\to\infty}\left(\frac{1}{x}\right)^2$$
$$=\lim_{x\to\infty}8-\left(\lim_{x\to\infty}\frac{1}{x}\right)^2=8-(0)^2=8.$$

2. $$\lim_{x\to\infty}\frac{1}{2+\frac{1}{x}}=\frac{\lim_{x\to\infty}1}{\lim_{x\to\infty}\left(2+\frac{1}{x}\right)}$$
$$=\frac{\lim_{x\to\infty}1}{\left(\lim_{x\to\infty}2\right)+\lim_{x\to\infty}\left(\frac{1}{x}\right)}$$
$$=\frac{1}{2+0}=\frac{1}{2}.$$

Limits at Infinity of Rational Functions

To determine the limit of a rational function as $x \to \pm\infty$, we first divide the numerator and denominator by the highest power of x in the denominator. Then take the limit as $x \to \pm\infty$. The result then depends on the degrees of the polynomials involved.

Example 3

Example

1.

$$\lim_{x\to\infty} \frac{2x+3}{5x+7} = \lim_{x\to\infty} \left(\frac{2x+3}{5x+7}\right) \cdot \frac{\frac{1}{x}}{\frac{1}{x}} = \lim_{x\to\infty} \left(\frac{2+\frac{3}{x}}{5+\frac{7}{x}}\right)$$

$$= \frac{\lim_{x\to\infty}\left(2+\frac{3}{x}\right)}{\lim_{x\to\infty}\left(5+\frac{7}{x}\right)}$$

$$= \frac{\lim_{x\to\infty} 2 + \lim_{x\to\infty} \frac{3}{x}}{\lim_{x\to\infty} 5 + \lim_{x\to\infty} \frac{7}{x}} = \frac{2+0}{5+0} = \frac{2}{5}.$$

Example 3

Example

2.

$$\lim_{x\to\infty}\frac{9x^3+x}{2x^4+5x^2-x+6}=\lim_{x\to\infty}\frac{9x^3+x}{2x^4+5x^2-x+6}\cdot\frac{1/x^4}{1/x^4}$$

$$=\lim_{x\to\infty}\frac{\frac{9}{x}+\frac{1}{x^3}}{2+\frac{5}{x^2}-\frac{1}{x^3}+\frac{6}{x^4}}$$

$$=\frac{\lim_{x\to\infty}\frac{9}{x}+\lim_{x\to\infty}\frac{1}{x^3}}{\lim_{x\to\infty}2+\lim_{x\to\infty}\frac{5}{x^2}-\lim_{x\to\infty}\frac{1}{x^3}+\lim_{x\to\infty}\frac{6}{x^4}}$$

$$=\frac{0+0}{2+0-0+0}=0.$$

Horizontal Asymptotes

If the distance between the graph of a function and some fixed line approaches zero as a point on the graph moves increasingly far from the origin, we say that the graph approaches the line asymptotically and that the line is an **asymptote** of the graph.

Definition

A line $y = b$ is a **horizontal asymptote** of the graph of a function $y = f(x)$ if either

$$\lim_{x\to\infty} f(x) = b \quad \text{or} \quad \lim_{x\to-\infty} f(x) = b.$$

Example 4

Example

Find the horizontal asymptotes, if any, of the function $f(x) = \arctan x$.

Solution

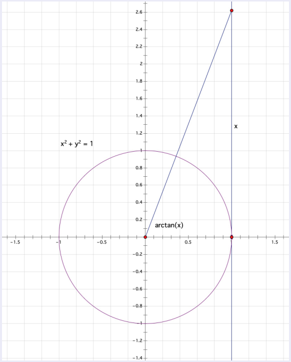


Figure: $\arctan(x)$ *for* x *positive*

To compute $\lim_{x\to\infty} \arctan(x)$*, we draw the unit circle with a line tangent to the circle at the point* $(1, 0)$*. If* x *is the distance from the* x*-axis upward along the tangent line, the labeled angle is then* $\arctan(x)$*.*

From the sketch in the figure, we see that if x *be comes very, very large, the angle* $\arctan(x)$ *goes to* $\pi/2$*.*

Solution

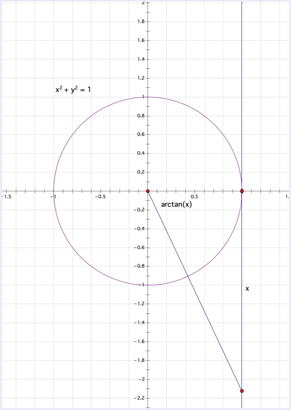


Figure: arctan(x) *for x negative*

To compute $\lim_{x \to -\infty} \arctan(x)$*, we draw the unit circle with a line tangent to the circle at the point* $(1, 0)$*. If x is minus the distance from the x-axis downward along the tangent line, the labeled angle is then* $\arctan(x)$*.*

From the sketch in the figure, we see that if x be comes very, very large, the angle $\arctan(x)$ *goes to* $-\pi/2$*.*

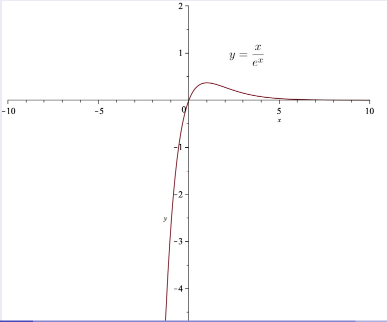
Example 5

Example

Find the horizontal asymptotes, if any, of the function $f(x) = \frac{x}{e^x}$.

Solution

We compute the limit of f as x approaches infinity using the graph of the function. From the graph, we see that x/e^x goes to zero as x goes to infinity.



Example 6

Example

Find **(a)** $\lim_{x\to\infty} \sin(1/x)$ and **(b)** $\lim_{x\to\infty} x\sin(1/x)$

Solution

a. *Let $t = 1/x$. Then*

$$\lim_{x\to\infty} \sin(1/x) = \lim_{t\to 0} \sin(t) = 0.$$

b. *Let $t = 1/x$. Then*

$$\lim_{x\to -\infty} x\sin(1/x) = \lim_{t\to 0^-} \frac{\sin(t)}{t} = 1.$$

Example 7

Example

Find $\lim_{x\to\infty}\left(x-\sqrt{x^2+4}\right)$.

Solution

We compute

$$\begin{aligned}
\lim_{x\to\infty}\left(x-\sqrt{x^2+4}\right) &= \lim_{x\to\infty}\left(x-\sqrt{x^2+4}\right)\cdot\frac{x+\sqrt{x^2+4}}{x+\sqrt{x^2+4}}\\
&= \lim_{x\to\infty}\frac{x^2-(x^2+4)}{x+\sqrt{x^2+4}}\\
&= \lim_{x\to\infty}\frac{-4}{x+\sqrt{x^2+4}}\\
&= 0.
\end{aligned}$$

Oblique Asymptotes

If the degree of the numerator of a rational function is 1 greater than the degree of the denominator, the graph has an **oblique asymptote**, a slanted line which the graph approaches as x goes to infinity.

We find an equation for the asymptote by dividing the numerator by the denominator to express f as a linear function plus a remainder that goes to zero as x goes to $\pm\infty$.

Example 8

Example

Find the oblique asymptote for the function

$$f(x) = \frac{x^3 + 1}{x^2}.$$

Solution

We do long division of polynomials:

$$\begin{array}{r|l} & \phantom{x^3+{}}x \\ \hline x^2 & x^3 + 1 \\ & -x^3 \\ \hline & \phantom{x^3+{}}1 \end{array}$$

From this, we see that

$$\frac{x^3 + 1}{x^2} = x + \frac{1}{x^2}.$$

Example 8

Solution

Since

$$\frac{x^3+1}{x^2} = x + \frac{1}{x^2}.$$

We have

$$\lim_{x\to\pm\infty} \frac{x^3+1}{x^2} - x = \lim_{x\to\pm\infty} \frac{1}{x^2} = 0.$$

We see that the function $f(x) = \dfrac{x^3+1}{x^2}$ *gets closer and closer to the function* $g(x) = x$ *for* x *very, very large.*

So, the oblique asymptote is $y = x$*.*

Infinite Limits

Suppose a rational function $f(x) = p(x)/q(x)$ where $p(x)$ and $q(x)$ are polynomials and suppose $x = c$ is a zero of q but not of p.

We can write $q(x) = (x - c)^m r(x)$, where m is the multiplicity of the root $x = c$ of $q(x)$ and $r(x)$ is a polynomial which doesn't vanish at $x = c$.

If we let $x \to c^+$, then $p(x)/r(x)$ goes to $p(c)/r(c)$, which is not zero by assumption, so for d near to c with $d > c$, $p(d)/q(d)$ is either positive or negative.

If $p(d)/q(d) > 0$, the function $f(x)$ goes to infinity as $x \to c^+$. If $p(d)/q(d) < 0$, the function $f(x)$ goes to minus infinity as $x \to c^+$.

Infinite Limits

Once again, write $q(x) = (x - c)^m r(x)$, where m is the multiplicity of the root $x = c$ of $q(x)$ and $r(x)$ is a polynomial which doesn't vanish at $x = c$.

If we let $x \to c^-$, then $p(x)/r(x)$ goes to $p(c)/r(c)$, which is not zero by assumption, so for d near to c with $d < c$, $p(d)/q(d)$ is either positive or negative.

If $p(d)/q(d) > 0$, the function $f(x)$ goes to infinity as $x \to c^-$. If $p(d)/q(d) < 0$, the function $f(x)$ goes to minus infinity as $x \to c^-$.

Example 9

a. The function $f(x) = \frac{x+1}{x-1}$ goes to infinity if $x \to 1^+$ and to minus infinity if $x \to 1^-$.
b. The function $f(x) = \frac{x+1}{(x-1)^2}$ goes to infinity if $x \to 1^+$ or $x \to 1^-$.

Precise Definitions of Infinite Limits

1. We say that $f(x)$ **approaches infinity as** x **approaches** c, and write

$$\lim_{x \to c} f(x) = \infty$$

if for every positive real number M there exists a corresponding $\delta > 0$ such that

$$f(x) > B \quad \text{whenever} \quad 0 < |x - c| < \delta.$$

2. We say that $f(x)$ **approaches negative infinity as** x **approaches** c, and write

$$\lim_{x \to c} f(x) = -\infty$$

if for every negative real number $-M$ there exists a corresponding $\delta > 0$ such that

$$f(x) < -M \quad \text{whenever} \quad 0 < |x - c| < \delta.$$

Example 10

Example

Prove that $\lim_{x \to 0} \frac{1}{x^2} = \infty$.

Solution

Let $M > 0$. *Choose* $\delta = 1/\sqrt{M}$. *Then for* $0 < |x| < \delta$, *we have*

$$\frac{1}{x^2} > \frac{1}{\delta^2} = M.$$

Vertical Asymptotes

If we consider the function $f(x) = 1/x$, the function becomes unbounded as x goes to 0.

If $x > 0$, $f(x) = 1/x$ is positive and

$$\lim_{x \to 0^+} \frac{1}{x} = \infty.$$

If $x < 0$, $f(x) = 1/x$ is negative and

$$\lim_{x \to 0^-} \frac{1}{x} = -\infty.$$

Vertical Asymptotes

Definition

A line $x = a$ is a **vertical asymptote** of the graph of the function $y = f(x)$ if either

$$\lim_{x \to a^+} f(x) = \pm\infty \quad \text{or} \quad \lim_{x \to a^-} f(x) = \pm\infty$$

Example 11

Example

Find the horizontal and vertical asymptotes of the curve $y = \dfrac{2x+3}{3x-2}$.

Solution

If we perform long division of polynomials, we see that

$$y = \frac{2x+3}{3x-2} = \frac{2}{3} + \frac{13/3}{3x-2}.$$

As $x \to \pm\infty$*, the fraction on the right goes to zero,* y *goes to* $\frac{2}{3}$*. So,* $y = \frac{2}{3}$ *is a horizontal asymptote.*

So, $x = \frac{2}{3}$ *is a vertical asymptote for the graph.*

Example 11

Example

Find the horizontal and vertical asymptotes of the curve $y = \dfrac{2x+3}{3x-2}$.

Solution

$$y = \frac{2x+3}{3x-2} = \frac{2}{3} + \frac{13/3}{3x-2}.$$

As $x \to \frac{2}{3}^-$, the denominator in the fraction on the right goes to zero through negative values, so the fraction goes to $-\infty$. So, $\lim_{x\to\frac{2}{3}^-} = -\infty$.

As $x \to \frac{2}{3}^+$, the denominator in the fraction on the right goes to zero through positive values, so the fraction goes to ∞. So, $\lim_{x\to\frac{2}{3}^+} = \infty$.

So, $x = \frac{2}{3}$ is a vertical asymptote for the graph.

Example 12

Example

Find the horizontal and vertical asymptotes of the curve $y = \dfrac{4}{4 - x^2}$.

Solution

First, as $x \to \pm\infty$, $4 - x^2$ goes to negative infinity, so y goes to 0. So, $y = 0$ is a horizontal asymptote.

Now, write the fraction as $\frac{4}{(2-x)(2+x)}$. We see the function is undefined at $x = \pm 2$.

To the left of -2, the numerator is positive, the first factor in the denominator goes to 4, and the second factor in the denominator goes to zero through negative values. So, the fraction is large and negative. So, $\lim_{x\to -2^-} \frac{4}{4-x^2} = -\infty$.

Example 12

Solution

To the right of -2*, the numerator is positive, the first factor in the denominator goes to* 4*, and the second factor in the denominator goes to zero through positive values. So, the fraction is large and positive. So,* $\lim_{x\to -2^+} \frac{4}{4-x^2} = \infty$.

To the left of 2*, the numerator is positive, the second factor in the denominator goes to* 4*, and the first factor in the denominator goes to zero through positive values. So, the fraction is large and positive. So,* $\lim_{x\to 2^-} \frac{4}{4-x^2} = \infty$.

To the right of 2*, the numerator is positive, the second factor in the denominator goes to* 4*, and the first factor in the denominator goes to zero through negative values. So, the fraction is large and negative. So,* $\lim_{x\to 2^+} \frac{4}{4-x^2} = -\infty$.

So, $x = \pm 2$ *are vertical asymptotes.*