

Continuity

William M. Faucette

University of West Georgia

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Continuity

- As usual, you should read section 2.5 in the online textbook.
- This slideshow will give an overview and an explanation of the important concepts in the book.
- This slideshow will also include a limited number of examples.
- The main purpose of this slideshow is to give an extended explanation and clarification of the material in the text.

Continuity at a Point

Intuitively, a function is continuous if the graph can be drawn without lifting your pencil from the paper. The graph has no holes or jumps or asymptotes.

Continuity is a concept that is defined locally, so we must start by defining what it means to be continuous at a point.

Continuity at a Point

Definition

Let c be a real number that is either an interior point or an endpoint of an interval in the domain of f .

- ① The function f is **continuous at c** if

$$\lim_{x \rightarrow c} f(x) = f(c).$$

- ② The function f is **right-continuous at c** if

$$\lim_{x \rightarrow c^+} f(x) = f(c).$$

- ③ The function f is **left-continuous at c** if

$$\lim_{x \rightarrow c^-} f(x) = f(c).$$

Continuity at a Point

The definition of (two-sided) continuity means three things:

- 1 $f(c)$ is defined. So c must be in the domain of f . $f(c)$ is *where the function is* at $x = c$.
- 2 $\lim_{x \rightarrow c} f(x) = L$ exists. L is *where the function should be* at $x = c$ based on the values of f near c .
- 3 $f(c)$ is L . This says the function *is where it should to be*. That is, there are no holes in the graph.

Continuity of Functions In The Large

Since continuity is a local concept, to define continuity of a function we must define what continuity means for the function as a whole.

- 1 A function $f(x)$ is **continuous on an open interval** (a, b) if it is continuous at each point $x = c$ in (a, b) .
- 2 A function $f(x)$ is **continuous on a closed interval** $[a, b]$ if it is continuous on (a, b) , continuous from the right at $x = a$, and continuous from the left at $x = b$.
- 3 A function $f(x)$ is **continuous** if it is continuous at each point of its domain.

Examples of Continuous Functions

Almost all the functions you know and love are continuous on their domains. This includes

- 1 Polynomial functions and rational functions
- 2 Trigonometric functions
- 3 Inverse Trigonometric functions
- 4 Exponential functions
- 5 Logarithmic functions

Example of a Discontinuous Function

The only function you are reasonably acquainted with that is not continuous on its domain is the greatest integer function. This function can be denoted $[x]$ or $\lfloor x \rfloor$ or $\text{int}(x)$. It is defined by

$\text{int}(x) =$ the greatest integer less than or equal to x .

Example of a Discontinuous Function

The graph of the greatest integer function is shown below.

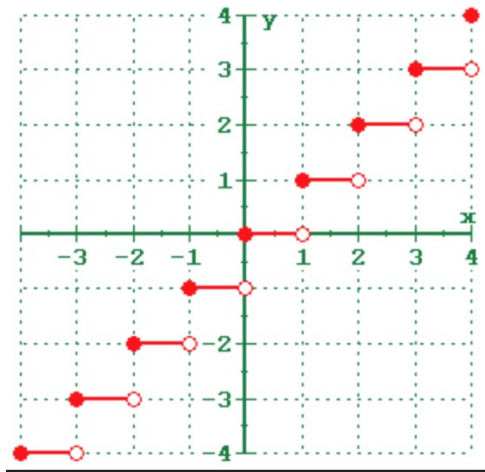


Figure: Graph of the Greatest Integer Function

Example of a Discontinuous Function

Looking at the graph of the greatest integer function, we can see that the function is discontinuous at every integer since there is a “jump” in the graph. For example, $\lim_{x \rightarrow 2^-} \text{int}(x) = 1$ and $\lim_{x \rightarrow 2^+} \text{int}(x) = 2$. So, the limit as x goes to 2 of $\text{int}(x)$ doesn't exist. The same is true at every integer: If $n \in \mathbb{Z}$ is any integer, then

$$\lim_{x \rightarrow n^-} \text{int}(x) = n - 1$$

$$\lim_{x \rightarrow n^+} \text{int}(x) = n$$

$$\text{int}(n) = n.$$

So, for each integer, $\text{int}(x)$ is continuous from the right, but not continuous.

Types of Discontinuities

- 1 If $f(x)$ is discontinuous at $x = c$, but $\lim_{x \rightarrow c} f(x)$ exists, then $x = c$ is a **removable discontinuity**. We can extend f to be continuous at $x = c$ by defining $f(c)$ to be the value of the limit $\lim_{x \rightarrow c} f(x)$.
- 2 If $f(x)$ is discontinuous at $x = c$ and the two one-sided limits exist and are not equal, then $x = c$ is a **jump discontinuity**.
- 3 A function such as $f(x) = \sin\left(\frac{1}{x}\right)$ has an **oscillating discontinuity** at $x = 0$.

Properties of Continuous Functions

If the functions f and g are continuous at $x = c$, then the following algebraic combinations are continuous at $x = c$.

1. Sums: $f + g$
2. Differences: $f - g$
3. Constant multiples: $k \cdot f$, for any number k
4. Products: $f \cdot g$
5. Quotients: f/g , provided $g(c) \neq 0$
6. Powers: f^n , n a positive integer
7. Roots: $\sqrt[n]{f}$, provided it is defined on an interval containing c , where n is a positive integer

Two Important Theorems

Theorem

Using the algebraic operations of scalar multiplication and addition/subtraction, we see that every polynomial

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$$

is continuous.

Theorem

If $P(x)$ and $Q(x)$ are polynomials, then the rational function $P(x)/Q(x)$ is continuous on its domain.

$$\frac{P(x)}{Q(x)} \text{ is continuous at } x = c \text{ provided } Q(c) \neq 0.$$

This follows from the previous theorem and the algebraic operation of division where the denominator is not zero.

Inverse Functions and Continuity

Theorem

If a function f is continuous and has an inverse function f^{-1} , then f^{-1} is continuous on its domain.

- 1 Since $f(x) = a^x$ is continuous, this theorem means $g(x) = \log_a(x)$ is continuous.
- 2 Since $f(x) = \sin x$ is continuous, this theorem means $g(x) = \sin^{-1}(x)$ is continuous.
- 3 Since $f(x) = \cos x$ is continuous, this theorem means $g(x) = \cos^{-1}(x)$ is continuous.

The same is true for the remaining trigonometric functions.

Compositions and Continuity

Recall that if $f(x)$ and $g(x)$ are functions, then their composition $(f \circ g)(x)$, defined by

$$(f \circ g)(x) = f(g(x)).$$

You take x , evaluate $g(x)$, and then put that result into f to get $f(g(x))$.

Theorem

If a function g is continuous at $x = c$ and f is continuous at $g(c)$, then the composition $f \circ g(x)$ is continuous at $x = c$.

That is, the composition of continuous functions is continuous.

Compositions and Continuity

- 1 $g(x) = x^2$ is continuous and $f(x) = \sin(x)$ is continuous, so $f \circ g(x) = \sin(x^2)$ is continuous.
- 2 $g(x) = \sqrt{x}$ is continuous and $f(x) = x^2 + 4$ is continuous, so $f \circ g(x) = \sqrt{x^2 + 4}$ is continuous.
- 3 $g(x) = \ln(x)$ is continuous and $f(x) = 2x^2 + 1$ is continuous, so $f \circ g(x) = \ln(2x^2 + 1)$ is continuous.

As you can see, by using algebraic operations on the functions we already know to be continuous, we can produce an enormous assortment of continuous functions.

Limits of Continuous Functions

If the inside function g in a composition has a limit at $x = c$ and the outside function f is continuous at that limit, then in the limit of the composition, the limit can be moved across the continuous function.

Theorem

If $\lim_{x \rightarrow c} g(x) = b$ and f is continuous at $x = b$, then

$$\lim_{x \rightarrow c} f(g(x)) = f(b).$$

That is

$$\lim_{x \rightarrow c} f(g(x)) = f(\lim_{x \rightarrow c} g(x)) = f(b).$$

Intermediate Value Theorem

The intermediate value theorem says if a function f takes on the value L at $x = a$ and M at $x = b$ and is continuous on the interval $[a, b]$, then f must assume every value between L and M on the interval $[a, b]$.

Theorem

If f is a continuous function on a closed interval $[a, b]$, and if y_0 is any value between $f(a)$ and $f(b)$, then $y_0 = f(c)$ for some c in $[a, b]$.

Intermediate Value Theorem

Referring to the figure below, if the graph of f passes through the point $(a, f(a))$ and $(b, f(b))$, and is continuous on $[a, b]$, then it must cross every horizontal line $y = y_0$ at some x value in the interval $[a, b]$.

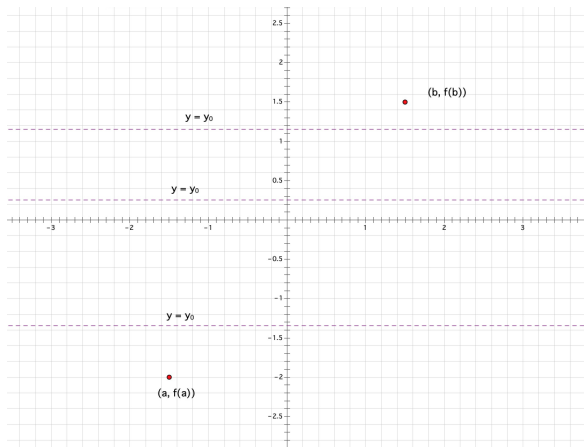


Figure: Sketch for the Intermediate Value Theorem

Intermediate Value Theorem

Probably the most common use of the Intermediate Value Theorem is in finding roots of functions. A continuous function that changes sign on an interval must have a root in that interval.

Theorem

If a function f is negative at $x = a$ and positive at $x = b$, $a < b$, and f is continuous on $[a, b]$, then f has a root in the interval $[a, b]$.

Similarly, if a function f is positive at $x = a$ and negative at $x = b$, $a < b$, and f is continuous on $[a, b]$, then f has a root in the interval $[a, b]$.