One-Sided Limits

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Outline

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One-Sided Limits

- As usual, you should read section 2.4 in the online textbook.
- This slideshow will give an overview and an explanation of the important concepts in the book.
- This slideshow will also include a limited number of examples.
- The main purpose of this slideshow is to give an extended explanation and clarification of the material in the text.

Approaching a Limit from One Side

In order for the limit as x approaches c of $f(x)$ to exist, the value of the limit cannot depend on the direction in which one approaches c. One can approach from the left of c, that is, through values of x less than c; or one can approach from the right of c, that is, through values of x greater than c. In order for the limit to exist, the value cannot depend on the direction in which x approaches c.

Approaching a Limit from One Side

If we only look at values of x approaching c from one side—either the left or the right—we get **one-sided limits**.

Intuitively, if x approaches c from the left, that is, through values of x less than c, and $f(x)$ approaches L, we say f has a **left-hand limit** (or **limit from the left**) of L at c.

If the left-hand limit of $f(x)$ as x approaches c is L, we write

$$\lim_{x \to c^-} f(x) = L.$$

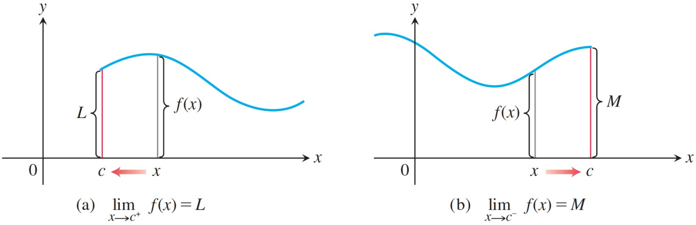
Approaching a Limit from One Side

Similarly, if x approaches c from the right, that is, through values of x greater than c, and $f(x)$ approaches L, we say f has a **right-hand limit** (or **limit from the right**) of L at c.

If the right-hand limit of $f(x)$ as x approaches c is M, we write

$$\lim_{x \to c^+} f(x) = M.$$

Approaching a Limit from One Side



(a) $\lim_{x \to c^+} f(x) = L$

(b) $\lim_{x \to c^-} f(x) = M$

FIGURE 2.25 (a) Right-hand limit as x approaches c. (b) Left-hand limit as x approaches c.

Figure: Visualization of Left- and Right-Hand Limits

Example 1

Let

$$f(x) = \begin{cases} -x+3 & \text{if } x < 2 \\ 2 & \text{if } x = 2 \\ 2x-1 & \text{if } x > 2. \end{cases}$$

Here's the graph of f:

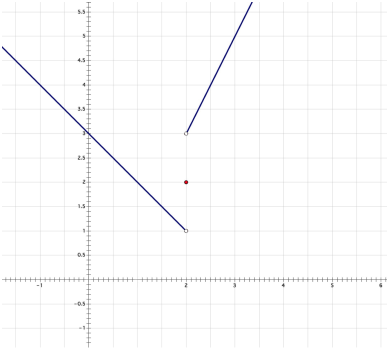


Figure: Graph of f

Example 1

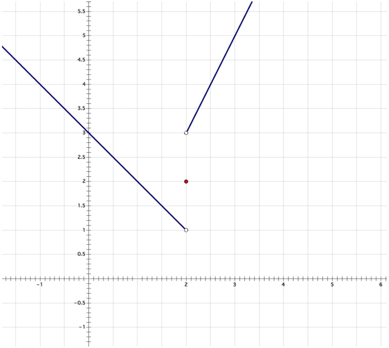


Figure: Graph of f

If you take $x < 2$, look at the point $(x, f(x))$ on the graph, and then let x move to the right toward 2, we see that $f(x)$ is approaching 1.
So,

$$\lim_{x \to 2^-} f(x) = 1.$$

Example 1

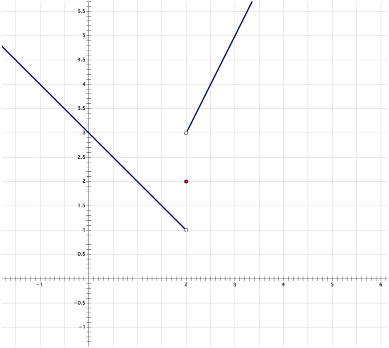


Figure: Graph of f

If you take $x > 2$, look at the point $(x, f(x))$ on the graph, and then let x move to the left toward 2, we see that $f(x)$ is approaching 3.
So,

$$\lim_{x \to 2^+} f(x) = 3.$$

Theorem

The two-sided limit of $f(x)$ as x approaches c is L exactly when the left- and right-hand limits exist and are both equal to L.

Theorem

Suppose that a function f is defined on an open interval containing c, except perhaps at c itself. Then $f(x)$ has a limit as x approaches c if and only if it has left-hand and right-hand limits there and these one-sided limits are equal.

$$\lim_{x\to c} f(x) = L \Longleftrightarrow \lim_{x\to c^-} f(x) = L \ \ \textit{and} \ \lim_{x\to c^+} f(x) = L$$

Precise Definitions of One-Sided Limits

The precise definitions of left and right-hand limits are very similar to the definition of the two-sided limit, except you restrict x to one side of c.

Definition

Let $f(x)$ be defined on an interval about (c,d) to the right of c. We say that the **right-hand limit of $f(x)$ as x approaches c is the number L**, and write

$$\lim_{x\to c^+} f(x) = L,$$

if, for every number $\epsilon > 0$, there exists a corresponding number $\delta > 0$ such that

$$|f(x) - L| < \epsilon \quad \text{whenever} \quad c < x < c + \delta.$$

Notice that x is restricted to being in the interval $(c, c+\delta)$, i.e. to the right of c.

Precise Definitions of One-Sided Limits

The precise definitions of left and right-hand limits are very similar to the definition of the two-sided limit, except you restrict x to one side of c.

Definition

Let $f(x)$ be defined on an interval about (b, c) to the left of c. We say that the **left-hand limit of $f(x)$ as x approaches c is the number L**, and write

$$\lim_{x \to c^-} f(x) = L,$$

if, for every number $\epsilon > 0$, there exists a corresponding number $\delta > 0$ such that

$$|f(x) - L| < \epsilon \quad \text{whenever} \quad c - \delta < x < c.$$

Notice that x is restricted to being in the interval $(c - \delta, c)$, i.e. to the left of c.

Example 2

Example

Prove that

$$\lim_{x \to 2^+} \sqrt{x-2} = 0.$$

Solution

Let $\epsilon > 0$. *Here* $c = 2$ *and* $L = 0$, *so we want to find a* $\delta > 0$ *such that*

$$|\sqrt{x-2} - 0| < \epsilon \textit{ whenever } 2 < x < 2 + \delta,$$

or

$$\sqrt{x-2} < \epsilon \textit{ whenever } 2 < x < 2 + \delta.$$

Example 2

Solution

We want $\delta > 0$ *so that*

$$\sqrt{x-2} < \epsilon \text{ whenever } 2 < x < 2+\delta.$$

Squaring both sides of the last inequality gives

$$x-2 < \epsilon^2 \text{ whenever } 2 < x < 2+\delta.$$

or

$$x-2 < \epsilon^2 \text{ whenever } 0 < x-2 < \delta.$$

If we choose $\delta = \epsilon^2$, *we have*

$$|\sqrt{x-2}| < \epsilon \text{ whenever } 2 < x < 2+\delta.$$

Example 3

Example

Show that $y = \sin(1/x)$ has no limit as x approaches zero from either side.

Solution

Let $x_n = \frac{1}{n\pi}$. *The sequence* (x_n) *goes to zero from the right and* $y_n = \sin(1/x_n) = \sin(n\pi) = 0$ *for all* n. *So, if* $\lim_{x\to 0^+} \sin(1/x)$ *exists, it must be 0.*

Example 3

Example

Show that $y = \sin(1/x)$ has no limit as x approaches zero from either side.

Solution

Let $x_n = \frac{2}{(4n-3)\pi}$. *The sequence* (x_n) *goes to zero from the right and* $y_n = \sin(1/x_n) = \sin((4n-3)\frac{\pi}{2}) = 1$ *for all* n. *So, if* $\lim_{x\to 0^+} \sin(1/x)$ *exists, it must be 1.*

Example 3

Example

Show that $y = \sin(1/x)$ has no limit as x approaches zero from either side.

Solution

These two results show that $\lim_{x\to 0^+} \sin(1/x)$ *does not exist.*

A similar argument shows that $\lim_{x\to 0^-} \sin(1/x)$ *does not exist.*

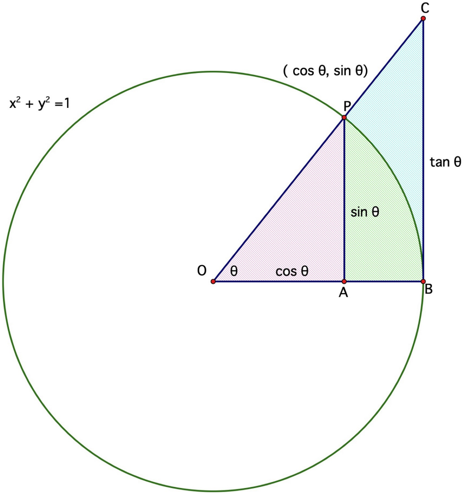
The Limit of $\dfrac{\sin\theta}{\theta}$

We want to examine

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta}.$$

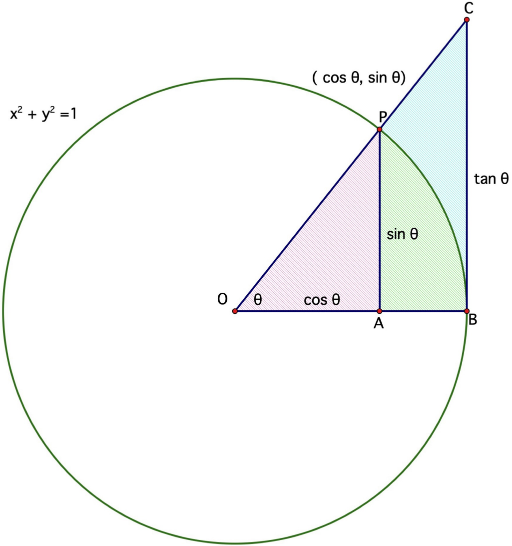
The Limit of $\dfrac{\sin\theta}{\theta}$

Consider the following sketch:

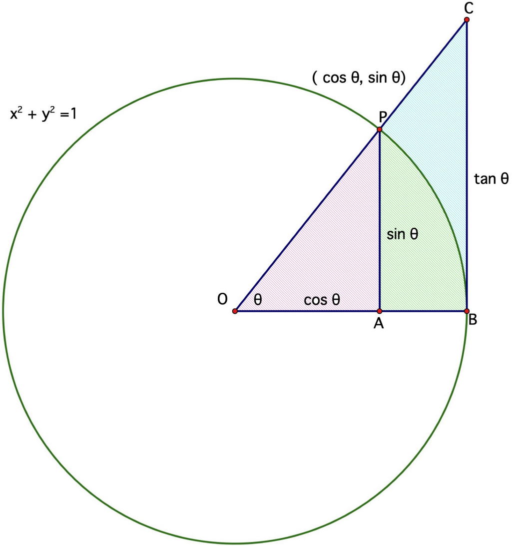


The Limit of $\dfrac{\sin\theta}{\theta}$

Notice the area of triangle ΔOAP is less than the area sector $\triangledown OBP$ which is less than the area of triangle ΔOBC.



The Limit of $\dfrac{\sin\theta}{\theta}$



The area of triangle ΔOAP is

$$\frac{1}{2}\cdot \text{base}\cdot\text{height}$$
$$= \frac{1}{2}\cos\theta\sin\theta.$$

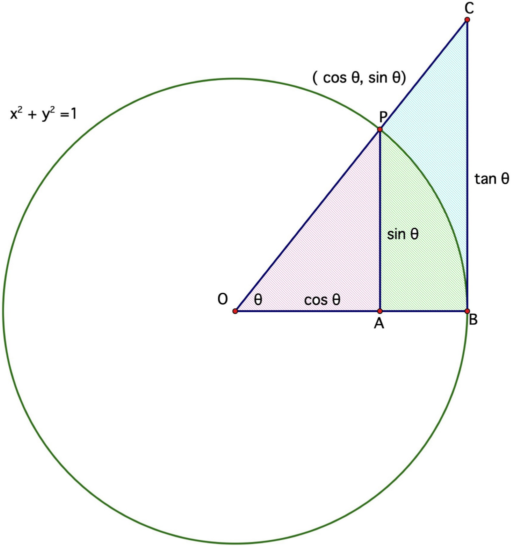
The Limit of $\dfrac{\sin\theta}{\theta}$

The area of sector $\triangledown OBP$ is

fraction of circle · area of circle

$$= \frac{\theta}{2\pi}\cdot\pi$$

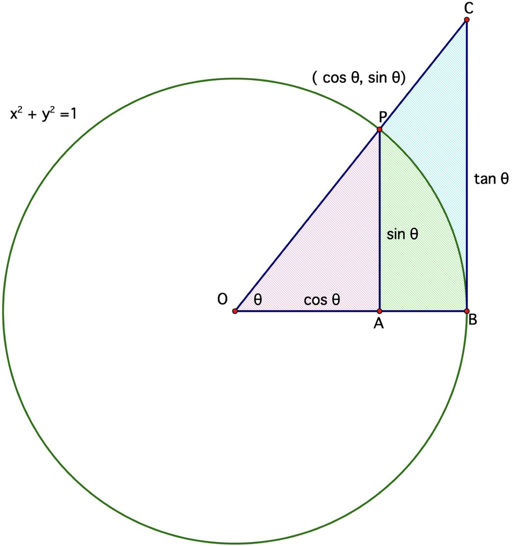
$$= \frac{1}{2}\theta.$$



The Limit of $\dfrac{\sin\theta}{\theta}$

The area of triangle ΔOBC is

$$\begin{aligned}\frac{1}{2}&\cdot \text{base}\cdot\text{height}\\ &=\frac{1}{2}(1)\tan\theta\\ &=\frac{1}{2}\tan\theta.\end{aligned}$$



The Limit of $\dfrac{\sin\theta}{\theta}$

This gives us

$$\frac{1}{2}\cos\theta\sin\theta < \frac{1}{2}\theta < \frac{1}{2}\tan\theta.$$

If we take θ slightly greater than zero and do some arithmetic, we get

$$\begin{gathered}\cos\theta\sin\theta < \theta < \frac{\sin\theta}{\cos\theta}\\ \cos\theta < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}.\end{gathered}$$

Taking reciprocals, we get

$$\cos\theta < \frac{\sin\theta}{\theta} < \frac{1}{\cos\theta}.$$

The Limit of $\dfrac{\sin\theta}{\theta}$

We previously had this inequality:

$$\frac{1}{2}\cos\theta\sin\theta < \frac{1}{2}\theta < \frac{1}{2}\tan\theta.$$

Similarly, if we take θ slightly less than zero and do some arithmetic, we get

$$\begin{gathered}\cos\theta\sin\theta < \theta < \frac{\sin\theta}{\cos\theta}\\ \cos\theta > \frac{\theta}{\sin\theta} > \frac{1}{\cos\theta}.\end{gathered}$$

Taking reciprocals, we get

$$\frac{1}{\cos\theta} < \frac{\sin\theta}{\theta} < \cos\theta.$$

The Limit of $\frac{\sin\theta}{\theta}$

In either case, the quantity

$$\frac{\sin\theta}{\theta}$$

lies between

$$\cos\theta \quad \text{and} \quad \frac{1}{\cos\theta}.$$

The Limit of $\dfrac{\sin\theta}{\theta}$

Since $\dfrac{\sin\theta}{\theta}$ lies between $\cos\theta$ and $\dfrac{1}{\cos\theta}$ and

$$\lim_{\theta\to 0}\cos\theta = \lim_{\theta\to 0}\frac{1}{\cos\theta} = 1,$$

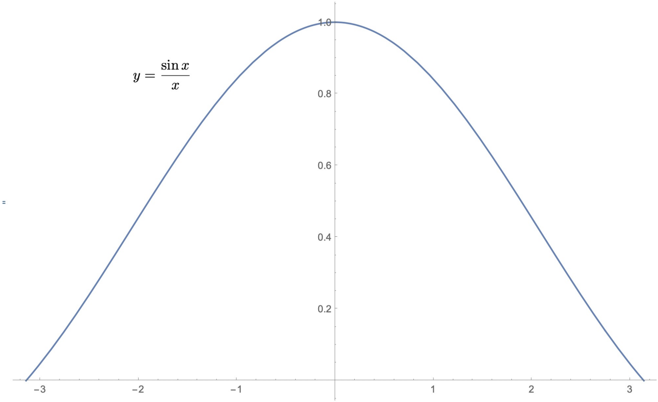
we have

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1.$$

by the Sandwich Theorem.

The Limit of $\dfrac{\sin\theta}{\theta}$

We can also see this by looking at the graph of the function $f(x) = \dfrac{\sin x}{x}$:



The Limit of $\dfrac{1 - \cos\theta}{\theta}$

There is one more limit we will need soon. That is the limit

$$\lim_{\theta \to 0} \frac{1 - \cos\theta}{\theta}.$$

The Limit of $\dfrac{\sin\theta}{\theta}$

To compute this limit, we do a bit of algebraic manipulation:

$$\begin{aligned}
\frac{1-\cos\theta}{\theta} &= \frac{1-\cos\theta}{\theta}\cdot\frac{1+\cos\theta}{1+\cos\theta} \\
&= \frac{1-\cos^2\theta}{\theta(1+\cos\theta)} \\
&= \frac{\sin^2\theta}{\theta(1+\cos\theta)} \\
&= \frac{\sin\theta}{\theta}\cdot\frac{\sin\theta}{1+\cos\theta}.
\end{aligned}$$

The Limit of $\dfrac{1-\cos\theta}{\theta}$

Taking the limit, we get

$$\begin{aligned}
\lim_{\theta\to 0}\frac{1-\cos\theta}{\theta} &= \lim_{\theta\to 0}\left[\frac{\sin\theta}{\theta}\cdot\frac{\sin\theta}{1+\cos\theta}\right] \\
&= \lim_{\theta\to 0}\frac{\sin\theta}{\theta}\cdot\lim_{\theta\to 0}\frac{\sin\theta}{1+\cos\theta} \\
&= 1\cdot 0 \\
&= 0.
\end{aligned}$$

Two New Limits

So, we have two new limits:

$$\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$$

and

$$\lim_{\theta \to 0} \frac{1 - \cos \theta}{\theta} = 0.$$