The Precise Definition of a Limit

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Outline

The Precise Definition of a Limit

- As usual, you should read section 2.3 in the online textbook.
- This slideshow will give an overview and an explanation of the important concepts in the book.
- This slideshow will also include a limited number of examples.
- The main purpose of this slideshow is to give an extended explanation and clarification of the material in the text.

Definition of a Limit

Mathematics is precise. It's based on established definitions and axioms and then uses logic to deduce results called theorems.

The idea of $f(x)$ getting "closer and closer" to L as x gets "closer and closer" to c is imprecise. It is warm, touchy, feely.

Mathematics is not warm, touchy, feely. Mathematics is cold and prickly. We need a precise definition of limit to do mathematics.

Definition of a Limit

Definition

Let $f(x)$ be defined on an open interval about c, except possibly at c itself. We say that the **limit of $f(x)$ as x approaches c is the number L**, and write

$$\lim_{x\to c} f(x) = L,$$

if, for every number $\epsilon > 0$, there exists a corresponding number $\delta > 0$ such that

$$|f(x) - L| < \epsilon \quad \text{whenever} \quad 0 < |x - c| < \delta.$$

What this means is this: If you give me a number $\epsilon > 0$ that is how close you want $f(x)$ to be from L, then I can give you a corresponding number $\delta > 0$ so that for all numbers x within a distance of δ from c, but $x \neq c$, then the value of $f(x)$ is within ϵ of L.

Example 1

Example

Show that

$$\lim_{x \to 2} (3x - 1) = 5.$$

Solution

Let $\epsilon > 0$ *be arbitrary. Let* $\delta = \epsilon/3$*. Suppose* $0 < |x - 2| < \delta$*. Then*

$$\begin{aligned} |f(x) - L| &= |(3x - 1) - 5| = |3x - 6| \\ &= |3(x - 2)| = 3|x - 2| \\ &< 3 \cdot \frac{\epsilon}{3} = \epsilon. \end{aligned}$$

This shows that for any $\epsilon > 0$, there exists $\delta > 0$—namely $\epsilon/3$—so that whenever $0 < |x - 2| < \delta$, the distance from $3x - 1$ to 5 is less that ϵ.

Finding Deltas Algebraically for Given Epsilons

How to Find Algebraically a δ for a given f, L, c, and $\epsilon > 0$

1. Solve the inequality $|f(x) - L| < \epsilon$ to find an open integral (a, b) containing c on which the inequality holds for all $x \neq c$. Note that we do not require the inequality to hold at $x = c$.
2. Find a value of $\delta > 0$ that places the open interval $(c - \delta, c + \delta)$ centered at c inside the interval (a, b). The inequality $|f(x) - L| < \epsilon$ will hold for all $x \neq c$ in this δ-interval.

Example 2

Example

For the limit $\lim_{x\to 13}\sqrt{x-4}=3$, find a $\delta>0$ that works for $\epsilon=1$.

Solution

We want $|\sqrt{x-4}-3|<1$. *We do a bit of algebra:*

$$
\begin{aligned}
|\sqrt{x-4}-3| &< 1\\
-1<\sqrt{x-4}-3 &< 1\\
2<\sqrt{x-4} &< 4\\
4=2^2<x-4 &< 4^2=16\\
8<x &< 20\\
8<13-4<x &< 13+4<20.
\end{aligned}
$$

So, we can take $\delta=4$. *As long as* $0<|x-13|<4$, *then* $|\sqrt{x-4}-3|<1$.

Using the Definition to Prove Theorems

The purpose for having a precise definition of limit is so we can prove theorems—such as the Limit Laws presented earlier—to facilitate computations of limits.

The fact that the limit of a constant is that constant, the limit of a sum is the sum of the limits, the limit of a difference is the difference of the limits, the limit of a product is the product of the limits, the limit of a quotient is the quotient of the limits (provided the denominator doesn't go to zero), etc., are all theorems which must be proven to be valid.

Example 3

Example

Given that $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$, prove that

$$\lim_{x\to c}(f(x) - g(x)) = L - M.$$

Solution

Let $\epsilon > 0$. *We want*

$$\begin{aligned} |f(x) - g(x) - (L - M)| &< \epsilon \\ |(f(x) - L) - (g(x) - M)| &< \epsilon \end{aligned}$$

We know we can make $f(x)$ *close to* L *and* $g(x)$ *close to* M. *This is the key.*

Using the Definition to Prove Theorems

Solution

Since $\lim_{x\to c} f(x) = L$*, we can choose* $\delta_1 > 0$ *so that whenever* $0 < |x - c| < \delta_1$, $|f(x) - L| < \epsilon/2$.

Since $\lim_{x\to c} g(x) = M$*, we can choose* $\delta_2 > 0$ *so that whenever* $0 < |x - c| < \delta_2$, $|g(x) - M| < \epsilon/2$.

Let δ *be the smaller of* δ_1 *and* δ_2.

Then whenever $0 < |x - c| < \delta$*, we have* $|f(x) - L| < \epsilon/2$ *and* $|g(x) - M| < \epsilon/2$.

Using the Definition to Prove Theorems

Solution

Let $0 < |x - c| < \delta$. *Then we have*

$$\begin{aligned} |f(x) - g(x) - (L - M)| &= |f(x) - M - (g(x - M)| \\ &\leq |f(x) - M| + |g(x - M)| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &< \epsilon. \end{aligned}$$

This proves the theorem.