Limit of a Function and Limit Laws

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Outline

Limit of a Function and Limit Laws

- As usual, you should read section 2.2 in the online textbook.
- This slideshow will give an overview and an explanation of the important concepts in the book.
- This slideshow will also include a limited number of examples.
- The main purpose of this slideshow is to give an extended explanation and clarification of the material in the text.

Limits of Function Values

The concept of a limit involves looking at the values of a function $f(x)$ **near** $x = c$, but not **at** $x = c$.

So, we want to evaluate the function $f(x)$ at points nearer and nearer $x = c$, but not $x = c$, and determine the behavior of the function there. We want to know the behavior of the function near $x = c$, not at it. The function may not even be defined at $x = c$.

Example 1

Example

How does the function

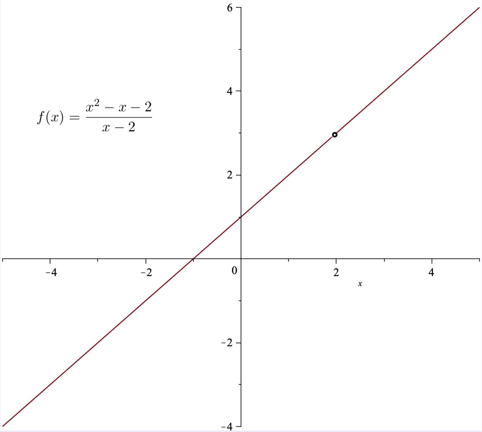
$$f(x) = \frac{x^2 - x - 2}{x - 2}$$

behave near $x = 2$?

Example 1

Solution

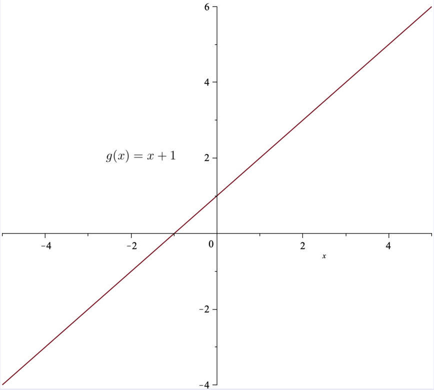
The function $f(x) = \dfrac{x^2 - x - 2}{x - 2}$ *is undefined at* $x = 2$*. We see the graph of this function has a hole at* $x = 2$*.*



Example 1

Solution

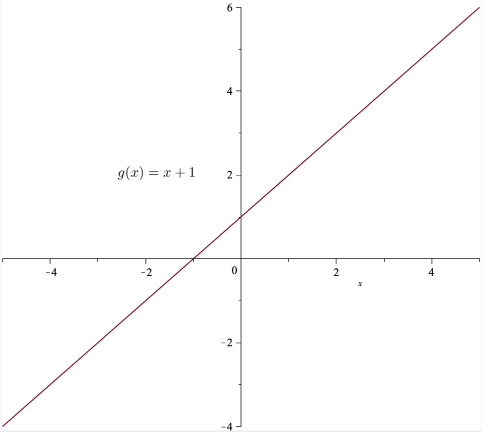
However $f(x) = \dfrac{x^2 - x - 2}{x - 2} = \dfrac{(x-2)(x+1)}{x-2} = x + 1$ *for all* $x \neq 2$, *the graph of* $f(x)$ *and the graph of* $g(x) = x + 1$ *are the same everywhere except* $x = 2$.



Example 1

Solution

Even though f is not defined at $x = 2$, $f(x) = x + 1$ for all $x \neq 2$, and it's clear we can make $x + 1$ as close to 3 as we want by choosing x sufficiently close to 2.



An Informal Description of the Limit of a Function

We start with a function defined on an open interval containing $x = c$, except possibly at $x = c$ itself.

If $f(x)$ can be made arbitrarily close to L by choosing x sufficiently close to c, then we say f approaches the limit L as x approaches c, and write

$$\lim_{x \to c} f(x) = L$$

which is read "the limit of $f(x)$ as x approaches c is L."

Example 2

Example

We saw in Example 1 that the function

$$f(x) = \frac{x^2 - x - 2}{x - 2}$$

gets closer and closer to 3 as x gets closer and closer to 2.
We write

$$\lim_{x \to 2} f(x) = \lim_{x \to 2} \frac{x^2 - x - 2}{x - 2} = 3.$$

Example 2

Solution

Consider instead the function

$$g(x) = \begin{cases} \dfrac{x^2 - x - 2}{x - 2}, & x \neq 2 \\ \quad 2 \quad, & x = 2. \end{cases}$$

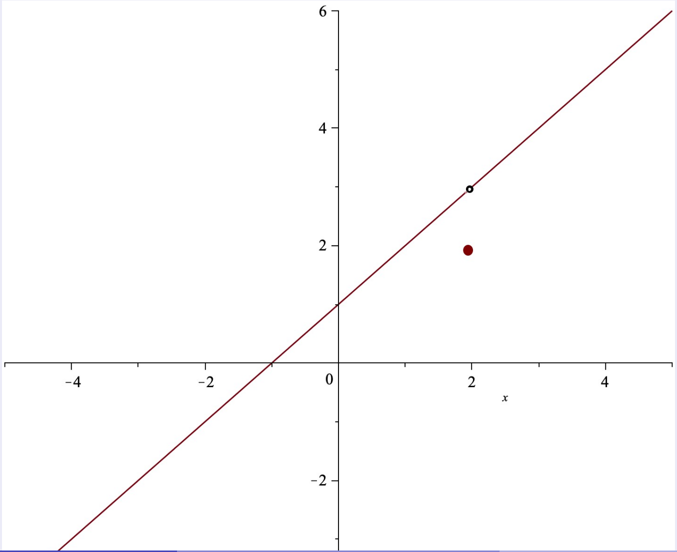
This is the function f *except we have defined the function at* $x = 2$ *to be* 2*. Since* $g(x) = f(x)$ *for all* $x \neq 2$*, it has the same limit as* $f(x)$*:*

$$\lim_{x \to 2} g(x) = 3.$$

(See the sketch on the next slide.)

Solution

We have defined f at $x = 2$, but we have defined it in the wrong place based on the values of the function near $x = 2$.



Example 2

Solution

Consider instead the function

$$h(x) = \begin{cases} \dfrac{x^2 - x - 2}{x - 2}, & x \neq 2 \\ 3 \quad , & x = 2. \end{cases}$$

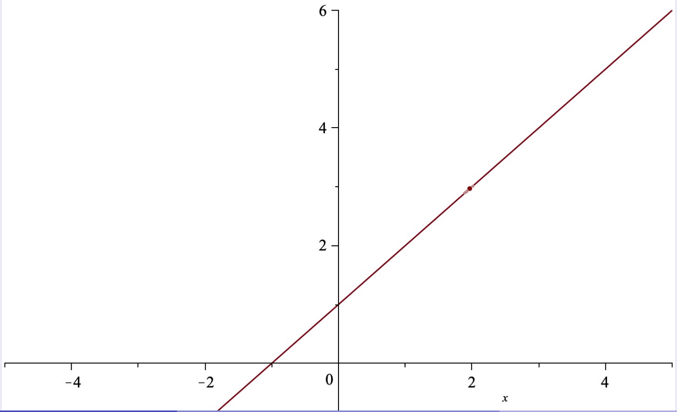
(See the sketch on the next slide.)

Solution

This is the function f except we have defined the function at $x = 2$ to be 3. Since $h(x) = f(x)$ for all $x \neq 2$, it has the same limit as $f(x)$:

$$\lim_{x\to 2} g(x) = 3.$$

We have defined f so that the hole at $x = 2$ is filled.



The Limit Laws

Theorem (The Limit Laws)

If $\lim_{x\to c} f(x) = L$ *and* $\lim_{x\to c} g(x) = M$ *for numbers* c, k, L, *and* M, *then*

Sum Rule: $\lim\limits_{x\to c}(f(x)+g(x)) = L+M$

Difference Rule: $\lim\limits_{x\to c}(f(x)-g(x)) = L-M$

Constant Multiple Rule: $\lim\limits_{x\to c}(kf(x)) = kL$

Product Rule: $\lim\limits_{x\to c}(f(x)g(x)) = LM$

Quotient Rule: $\lim\limits_{x\to c}\dfrac{f(x)}{g(x)} = \dfrac{L}{M}$, *if* $M \neq 0$

Power Rule: $\lim\limits_{x\to c}[f(x)]^n = L^n$

Root Rule: $\lim\limits_{x\to c}\sqrt[n]{f(x)} = \sqrt[n]{L}$, *if* n *is a positive integer*

(In the last rule if n *is even,* $f(x)$ *must be nonnegative near* c*.)*

Example 3

Example

$$\begin{aligned}
\lim_{x\to 2}(-x^2+5x-2) &= \lim_{x\to 2} -x^2 + \lim_{x\to 2} 5x - \lim_{x\to 2} 2 \\
&= -\lim_{x\to 2} x^2 + 5\lim_{x\to 2} x - \lim_{x\to 2} 2 \\
&= -\left(\lim_{x\to 2} x\right)^2 + 5\lim_{x\to 2} x - \lim_{x\to 2} 2 \\
&= -2^2 + 5\cdot 2 - 2 \\
&= 4.
\end{aligned}$$

Example 4

Example

$$\begin{aligned}
\lim_{y\to 2}\frac{y+2}{y^2+5y+6} &= \frac{\lim_{y\to 2}(y+2)}{\lim_{y\to 2}(y^2+5y+6)}\\
&= \frac{\lim_{y\to 2} y+\lim_{y\to 2} 2}{\lim_{y\to 2}(y^2)+\lim_{y\to 2}(5y)+\lim_{y\to 2} 6}\\
&= \frac{\lim_{y\to 2} y+\lim_{y\to 2} 2}{\left(\lim_{y\to 2} y\right)^2+5\lim_{y\to 2} y+\lim_{y\to 2} 6}\\
&= \frac{2+2}{(2)^2+5\cdot 2+6}\\
&= \frac{4}{20}=\frac{1}{5}.
\end{aligned}$$

Example 5

Example

$$\begin{aligned}
\lim_{h\to 0}\frac{\sqrt{5h+4}-2}{h} &= \lim_{h\to 0}\frac{\sqrt{5h+4}-2}{h}\cdot\frac{\sqrt{5h+4}+2}{\sqrt{5h+4}+2}\\
&= \lim_{h\to 0}\frac{(5h+4)-4}{h\left(\sqrt{5h+4}+2\right)} = \lim_{h\to 0}\frac{5{\color{red}h}}{{\color{red}h}\left(\sqrt{5h+4}+2\right)}\\
&= \lim_{h\to 0}\frac{5}{\sqrt{5h+4}+2} = \frac{\lim_{h\to 0}5}{\lim_{h\to 0}\left(\sqrt{5h+4}+2\right)}\\
&= \frac{\lim_{h\to 0}5}{\lim_{h\to 0}\sqrt{(5h+4)}+\lim_{h\to 0}2}\\
&= \frac{\lim_{h\to 0}5}{\sqrt{\lim_{h\to 0}(5h+4)}+\lim_{h\to 0}2}\\
&= \frac{5}{\sqrt{4}+2} = \frac{5}{4}.
\end{aligned}$$

Example 5

Example

Notice in this example, the fraction

$$\frac{\sqrt{5h+4}-2}{h}$$

is not defined at $h = 0$, but the limit as h goes to 0 exists.

Evaluating Limits of Polynomials and Rational Functions

Using the Laws of Limits repeatedly, it is easy to show the following:

Theorem (Limits of Polynomials)

If $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$*, then*

$$\lim_{x \to c} P(x) = a_n c^n + a_{n-1}c^{n-1} + \cdots + a_0 = P(c).$$

Theorem (Limits of Rational Functions)

If $P(x)$ *and* $Q(x)$ *are polynomials and* $Q(c) \neq 0$*, then*

$$\lim_{x \to c} \frac{P(x)}{Q(x)} = \frac{P(c)}{Q(c)}.$$

Example 6

Example

$$\lim_{x\to 2}\frac{2x+5}{11-x^2}=\frac{2\cdot 2+5}{11-2^2}=\frac{9}{7}.$$

Eliminating Common Factors from Zero Denominators

The theorem on the limit of rational functions only applies if the denominator is not zero at $x = c$.

If both the numerator and denominator of a fraction go to 0 as x goes to c, you need to find the common factor in the numerator and denominator that is going to 0 and cancel it.

Then you can apply the theorem on the limit of rational functions.

Example 7

Example

In this example, both the numerator and the denominator are 0 at $x = c$. We have to factor and cancel the common factor going to 0. Then we can apply the theorem on the limit of rational functions.

$$\begin{aligned}\lim_{x \to -3} \frac{x+3}{x^2+4x+3} &= \lim_{x \to -3} \frac{x+3}{(x+3)(x+1)} \\ &= \lim_{x \to -3} \frac{1}{x+1} \\ &= \frac{1}{(-3)+1} = -\frac{1}{2}.\end{aligned}$$

Using Calculators to Estimate Limits

You can use the table function on your calculator to approximate the limit of a function.

1. Press y= and enter the function.
2. Press 2nd tblset .
3. In the line that says Indpnt:, use the arrow keys to choose Ask. Press enter .
4. In the line that says Depend:, use the arrow keys to choose Auto. Press enter .
5. Press 2nd table .
6. One by one, enter numbers getting closer and closer to c, pressing enter after each entry.
7. The value of the function at that input will be computed and displayed in the Y column.
8. The number the sequence of y-values is getting close to is the limit.

Example 8

Example

Evaluate

$$\lim_{x \to 0} \frac{\sqrt{4+x}-2}{x}$$

Solution

We put the function

$$f(x) = \frac{\sqrt{4+x}-2}{x}$$

into the calculator.

Follow the steps on the preceding slide.

Example 8

Solution

If we input the following values for x*, this is the table we produce:*

x	y
1	0.2361
0.1	0.2485
0.01	0.2498
0.001	0.25
0.0001	0.25

From this we see that the limit is $1/4$.

The Sandwich Theorem

The Sandwich Theorem (or Squeeze Theorem) is used by finding two functions g and h which go to the same limit as x goes to c, one of which is smaller than f and one of which is larger than f. Since both g and h go to the same limit and f is trapped in between, it must also go to the same limit as well.

Theorem (The Sandwich Theorem)

Suppose that $g(x) \leq f(x) \leq h(x)$ *for all* x *in some open interval containing* c*, except possibly at* $x = c$*. Suppose also that*

$$\lim_{x \to c} g(x) = \lim_{x \to c} h(x) = L.$$

Then $\lim_{x \to c} f(x) = L$*.*

Example 9

Example

The values of the sine function are always between -1 and 1, so

$$-1 \leq \sin\left(\frac{1}{x}\right) \leq 1$$

for all values of $x \neq 0$. Multiplying by x^2, we get

$$-x^2 \leq x^2 \sin\left(\frac{1}{x}\right) \leq x^2.$$

If $g(x) = -x^2$ and $h(x) = x^2$, then both h and g go to zero as x goes to zero. So, we conclude that

$$\lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right) = 0.$$