Rates of Change and Tangent Lines to Curves

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Outline

Rates of Change and Tangent Lines to Curves

- As usual, you should read section 2.1 in the online textbook.
- This slideshow will give an overview and an explanation of the important concepts in the book.
- This slideshow will also include a limited number of examples.
- The main purpose of this slideshow is to give an extended explanation and clarification of the material in the text.

Average and Instantaneous Speed

Suppose a moving object has traveled a distance $f(t)$ at time t. The average speed during the time interval $[t_1, t_2]$ is the change in distance divided by the change in time. The unit of measure is length per unit time.

Average Speed

When $f(t)$ measures the distance traveled at time t, the average speed over the interval $[t_1, t_2]$ is

$$\frac{f(t_2) - f(t_1)}{t_2 - t_1}.$$

Example 1

Example

A solid object dropped from rest near the surface of the earth and allowed to fall freely will fall according to the following equation:

$$s(t) = 16t^2$$

where t is measured in seconds and $s(t)$ is measured in feet. What is the object's average speed during the first 2 sec of fall? During the time between $t = 1$ sec and $t = 2$ sec.

Example 1

Solution

The object's average speed during the first 2 sec of fall is

$$\frac{s(2)-s(0)}{2-0}=\frac{16(2)^2-16(0)^2}{2-0}=\frac{64}{2}=32\ ft/s.$$

The object's average speed during the time between $t=1$ *sec and* $t=2$ *sec.*

$$\begin{aligned}\frac{s(2)-s(1)}{2-1}&=\frac{16(2)^2-16(1)^2}{2-1}\\&=\frac{64-16}{2-1}=48\ ft/s.\end{aligned}$$

Example 2

Example

A solid object dropped from rest near the surface of the earth and allowed to fall freely will fall according to the following equation:

$$s(t) = 16t^2$$

where t is measured in seconds and $s(t)$ is measured in feet.
Find the speed of the falling rock in Example 1 between time $t = 1$ and $t = 1 + h$.

Example 2

Solution

The speed of the falling rock in Example 1 between time $t = 1$ and $t = 1 + h$ is

$$\begin{aligned}
\frac{s(1+h) - s(1)}{(1+h) - 1} &= \frac{16(1+h)^2 - 16(1)^2}{(1+h) - 1} \\
&= \frac{16(1 + 2h + h^2) - 16}{h} \\
&= \frac{16 + 32h + 16h^2 - 16}{h} \\
&= \frac{32h + 16h^2}{h} \\
&= \frac{h(32 + 16h)}{h} \\
&= 32 + 16h.
\end{aligned}$$

Example 2

Solution

For values of h getting closer and closer to zero, we get

h	Average speed on $[1, 1 + h]$
1	48
0.1	33.6
0.01	32.16
0.001	32.016
0.0001	32.0016

Average Rates of Change and Secant Lines

Definition

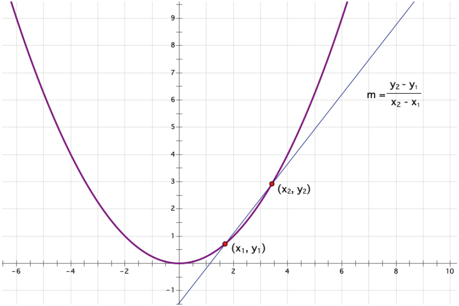
The **average rate of change** of $y = f(x)$ with respect to x over the interval $[x_1, x_2]$ is

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{f(x_1 + h) - f(x_1)}{h}, \quad h \neq 0.$$

Average Rates of Change and Secant Lines

On the graph $y = f(x)$, the average rate of change over the interval $[x_1, x_2]$ is the slope of the line between the points $(x_1, f(x_1))$ and $(x_2, f(x_2))$. This is a **secant line** to the graph.

Figure: Sketch of a Secant Line



Average Rates of Change and Secant Lines

A tangent line to a curve meets the curve at only one point.

In order to find the slope of the tangent line, we look at the slope of a secant line from a fixed point P_0 on the curve to a variable point P on the curve.

Then we will let the point P get closer and closer to P_0 and observe what happens to the slopes.

Example 3

Example

Find the slope of the tangent line to the parabola $y = f(x) = x^2 - x$ at the point $P_0(3, 6)$ by analyzing slopes of secant lines through $P_0(3, 6)$. Write an equation for the tangent line to the parabola at this point.

Example 3

Solution

We compute the slopes of secant lines through $P_0(3,6)$ *and* $P(3+h, f(3+h))$.

$$\begin{aligned}
\frac{f(3+h)-f(3)}{h} &= \frac{[(3+h)^2-(3+h)]-[3^2-3]}{h} \\
&= \frac{[9+6h+h^2-(3+h)]-[6]}{h} \\
&= \frac{6+5h+h^2-6}{h} \\
&= \frac{h(5+h)}{h} \\
&= 5+h.
\end{aligned}$$

Example 3

Solution

We form a table for values of h getting closer and closer to zero. This corresponds to the point P getting closer and closer to the point P_0.

h	Slope of Secant Line
1	6
0.1	5.1
0.01	5.01
0.001	5.001
0.0001	5.0001

As h gets closer and closer to zero, P is getting closer and closer to the point P_0. The secant lines between P_0 and P get closer and closer to a tangent line at P_0. So, the slopes of the secant lines between P_0 and P get closer and closer to the slope of the tangent line at P_0. Looking at the table, we see that the slopes of the secants lines are getting closer and closer to 5. So, the slope of the tangent line is 5.

Example 3

Solution

The equation of the tangent line to the curve $y = f(x) = x^2 - x$ *at the point* $(3, 6)$ *is*

$$\begin{aligned} y - 6 &= 5(x - 3) \\ y - 6 &= 5x - 15 \\ y &= 5x - 9. \end{aligned}$$

Rates of Change and Tangent Lines

The average rate of change of f on the interval $[x, x+h]$ is defined to be

$$\frac{f(x+h)-f(x)}{h}.$$

As h gets closer and closer to zero, the average rate of change of the function f on the interval $[x, x+h]$ gets closer and closer to the actual rate of change of the function f at x. This is the **instantaneous rate of change** of f at x.

Rates of Change and Tangent Lines

The instantaneous rate of change of a function f at $x = a$ is the slope of the tangent line to the graph $y = f(x)$ at the point $(a, f(a))$.

First, we have to make precise what it means for the average values of a function to get "closer and closer" to something as h, the length of the interval, gets "closer and closer" to zero.

This involves the concept of a **limit**.