Definite Integral Substitutions and the Area Between Curves

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Outline

Indefinite Integrals and the Substitution Method

- As usual, you should read section 5.6 in the online textbook.
- This slideshow will give an overview and an explanation of the important concepts in the book.
- This slideshow will also include a limited number of examples.
- The main purpose of this slideshow is to give an extended explanation and clarification of the material in the text.

Substitution in Definite Integrals

Fortunately, we have done almost all the work we have to do here in the last section.

When you make a u-substitution in a definite integral, just as in the last section, you substitute u's for x's, du's for dx's, **and** you have to change the **limits**.

Example 1

Example

Evaluate the definite integral

$$\int_0^{\pi} 3\cos^2 x \sin x \, dx.$$

Solution

Let $u = \cos x$. *Then* $du = -\sin x \, dx$ *and* $-du = \sin \, dx$. *This integral goes from* $x = 0$ *to* $x = \pi$. *When* $x = 0$, $u = 1$. *When* $x = \pi$, $u = -1$. *Substituting, we get*

$$\begin{aligned} \int_0^{\pi} 3\cos^2 x \sin x \, dx &= \int_1^{-1} -3u^2 \, du \\ &= -u^3 \Big|_1^{-1} \\ &= -(-1)^3 - (-(1)^3) = 2. \end{aligned}$$

Example 2

Example

Evaluate the definite integral

$$\int_1^9 \sqrt{4+5x}\,dx.$$

Example 2

Solution

Let $u = 4 + 5x$*. Then* $du = 5\,dx$ *and* $\frac{1}{5}du = dx$*. When* $x = 1$, $u = 9$*. When* $x = 9$, $u = 49$*. Substituting, we get*

$$\begin{aligned}
\int_1^9 \sqrt{4+5x}\,dx &= \int_9^{49} \sqrt{u} \cdot \frac{1}{5}\,du \\
&= \frac{1}{5}\int_9^{49} \sqrt{u}\,du \\
&= \frac{1}{5}\left[\frac{2}{3}u^{3/2}\right|_9^{49} \\
&= \frac{2}{15}\left(49^{3/2} - 9^{3/2}\right) \\
&= \frac{632}{15}.
\end{aligned}$$

Definite Integrals of Symmetric Functions

Recall the definition of an even function and an odd function.

Definition

A function is an **even function** if $f(-x) = f(x)$ for all x in the domain.

A function is an **odd function** if $f(-x) = -f(x)$ for all x in the domain.

These names come from the fact that the functions

$$1, x^2, x^4, x^6, \ldots \qquad \text{are even functions}$$

and

$$x, x^3, x^5, x^7, \ldots \qquad \text{are odd functions.}$$

Definite Integrals of Symmetric Functions

Recall also . . .

Symmetries of Odd and Even Functions

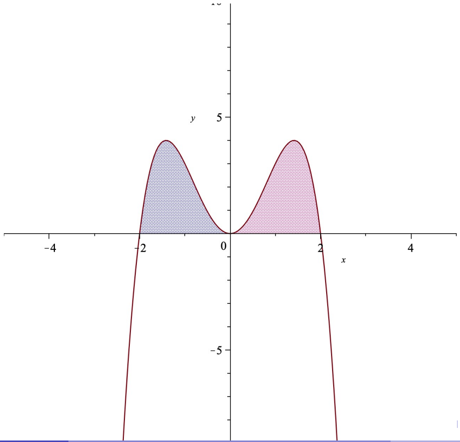
The graph of an odd function is symmetric about the origin.

The graph of an even function is symmetric about the y-axis.

Definite Integrals of Symmetric Functions

If you integrate an even function from $-a$ to a, the value of the integral from $-a$ to 0 is equal to the value of the integral from 0 to a. So, the entire integral from $-a$ to a is twice the integral from 0 to a.

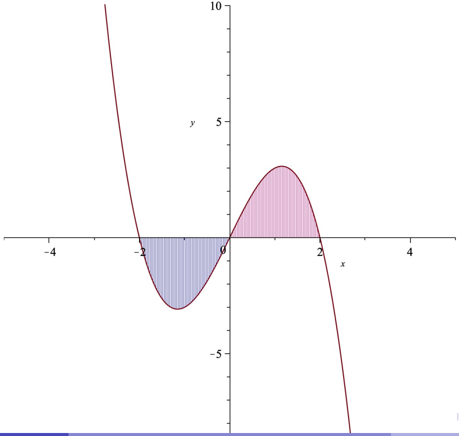
Figure: Symmetry of an Even Function



Definite Integrals of Symmetric Functions

If you integrate an odd function from $-a$ to a, the value of the integral from $-a$ to 0 is equal—but opposite in sign—to the value of the integral from 0 to a. So, the entire integral from $-a$ to a is 0.

Figure: Symmetry of an odd Function



Definite Integrals of Symmetric Functions

This gives us the following result:

Theorem

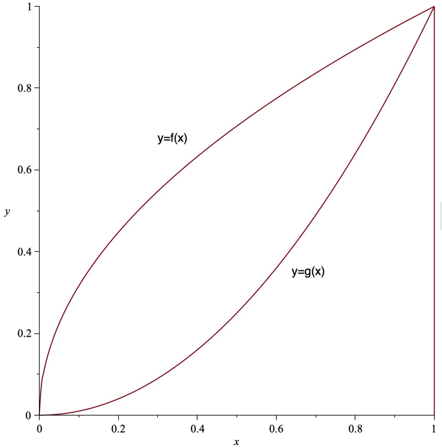
Let f be continuous on the symmetric interval $[-a, a]$.

1. *If f is even, then $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$.*
2. *If f is odd, then $\int_{-a}^{a} f(x)\,dx = 0$.*

Area Between Curves

Suppose you have two functions f and g defined on an interval $[a, b]$ so that $f(x) \geq g(x)$ for all x in $[a, b]$.

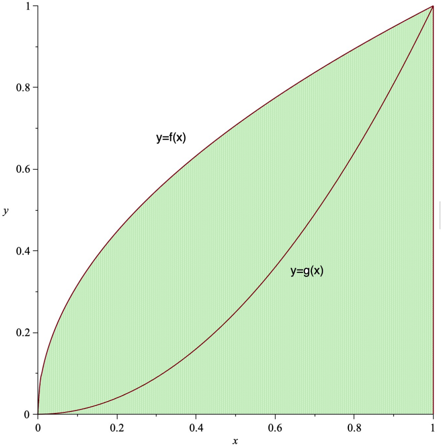
Figure: Graph of $f(x) \geq g(x)$ on $[0, 1]$



Area Between Curves

The integral $\int_a^b f(x)\,dx$ gives you the area under $y = f(x)$ between $x = a$ and $x = b$. This is the region shaded green in the figure below.

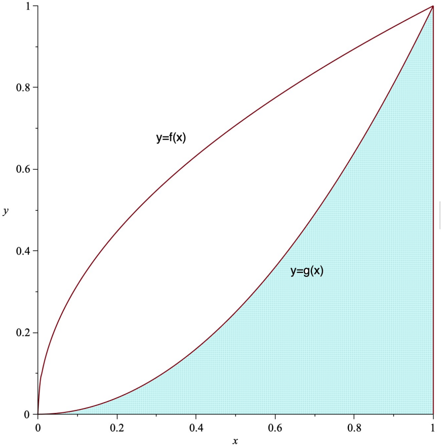
Figure: Area Under $y = f(x)$



Area Between Curves

The integral $\int_a^b g(x)\,dx$ gives you the area under $y = g(x)$ between $x = a$ and $x = b$. This is the region shaded light blue in the figure below.

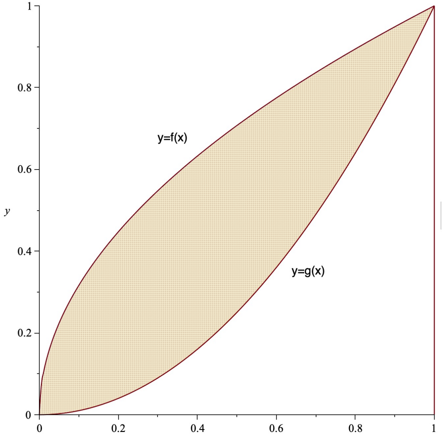
Figure: Area Under $y = g(x)$



Area Between Curves

The difference between these two, gives you the area between $y = f(x)$ and $y = g(x)$ between $x = a$ and $x = b$. This is the region shaded beige in the figure below.

Figure: Area Between $y = f(x)$ and $y = g(x)$



Area Between Curves

Definition

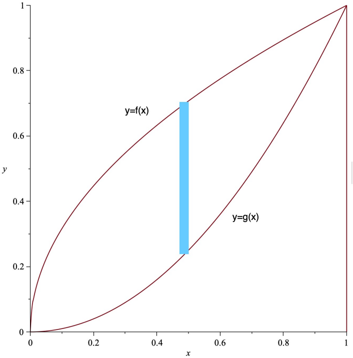
If f and g are continuous with $f(x) \geq g(x)$ throughout $[a, b]$, then the **area of the region between the curves** $y = f(x)$ **and** $y = g(x)$ **from** a **to** b is the integral of $(f - g)$ from a to b:

$$A = \int_a^b [f(x) - g(x)]\, dx.$$

Area Between Curves

An important way to understand this is given in the following picture. The width of this rectangle can be thought of as dx. The height of the rectangle can be approximated by $f(x) - g(x)$. So its area can be thought of as $[f(x) - g(x)]\ dx$. Adding these by integrating gives you the area between the curves.

Figure: Rectangle for Riemann Sum



Example 3

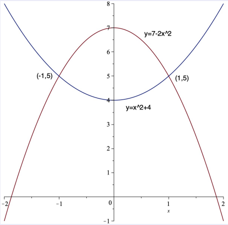
Example

Find the area enclosed between the curves $y = 7 - 2x^2$ and $y = x^2 + 4$.

Solution

First, we plot the curves and look at the region. A little algebra shows that the curves meet at the points $(-1, 5)$ *and* $(1, 5)$*.*

Figure: *Sketch for Example 3*



Example 3

Solution

From the sketch of the region, we see that the area between the two curves is given by

$$\begin{aligned}\int_{-1}^{1} [f(x) - g(x)]\, dx &= \int_{-1}^{1} \left[(7 - 2x^2) - (x^2 + 4)\right]\, dx \\ &= \int_{-1}^{1} -3x^2 + 3\, dx \\ &= 2\int_{0}^{1} -3x^2 + 3\, dx \text{ since the integrand is even} \\ &= 2\left[-x^3 + 3x\right|_0^1\, dx \\ &= 2\left[(-(1)^3 + 3(1)) - 0\right] \\ &= 4.\end{aligned}$$

Area Between Curves

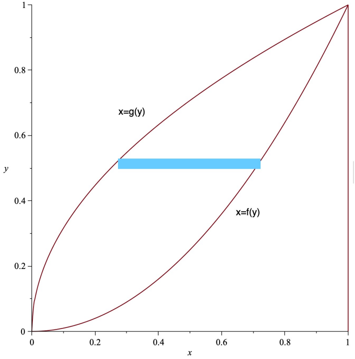
If the curves are given as $x = f(y)$ and $x = g(y)$, then the approximating rectangles are horizontal instead of vertical and the basic formula has y in place of x. For regions like these, the area between the curves is

$$A = \int_c^d [f(y) - g(y)]\, dy.$$

Area Between Curves

The height of this rectangle can be thought of as dy. The width of the rectangle can be approximated by $f(y) - g(y)$. So its area can be thought of as $[f(y) - g(y)]\, dy$. Adding these by integrating give you the area between the curves.

Figure: Rectangle for Riemann Sum



Example 4

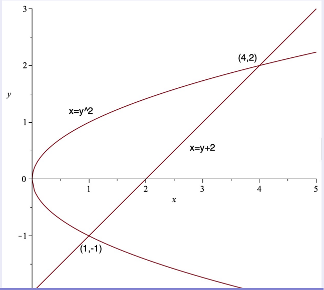
Example

Find the area enclosed between the curves $x = y^2$ and $x = y + 2$.

Solution

First, we plot the curves and look at the region. A little algebra shows that the curves meet at the points $(1, -1)$ *and* $(4, 2)$*.*

Figure: *Sketch for Example 4*



Example 4

Solution

From the sketch of the region, we see that the area between the two curves is given by

$$\begin{aligned}
\int_{-1}^{2} [f(y) - g(y)]\ dy &= \int_{-1}^{2} \left[(y+2) - (y^2)\right]\ dy \\
&= \int_{-1}^{2} -y^2 + y + 2\, dy \\
&= -\frac{1}{3}y^3 + \frac{1}{2}y^2 + 2y\Big|_{-1}^{2}\ dy \\
&= \left(-\frac{1}{3}(2)^3 + \frac{1}{2}(2)^2 + 2(2)\right) \\
&\qquad - \left(-\frac{1}{3}(-1)^3 + \frac{1}{2}(-1)^2 + 2(-1)\right) \\
&= \frac{9}{2}.
\end{aligned}$$