Area and Estimating with Finite Sums

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Outline

1. General Instructions
2. Area and Estimating with Finite Sums
3. Area Under a Curve
4. Distance Traveled
5. Displacement versus Distance Traveled

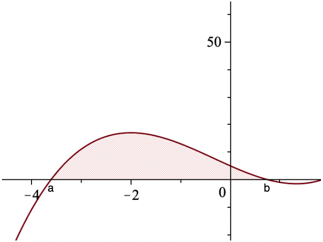
Area and Estimating with Finite Sums

- As usual, you should read section 5.1 in the online textbook.
- This slideshow will give an overview and an explanation of the important concepts in the book.
- This slideshow will also include a limited number of examples.
- The main purpose of this slideshow is to give an extended explanation and clarification of the material in the text.

Area and Estimating with Finite Sums

Suppose we have a function $f(x)$ which is non-negative on the interval $[a, b]$. How do we find the area under the graph $y = f(x)$ over the interval $[a, b]$?

Figure: Graph of $y = f(x)$ on $[a, b]$



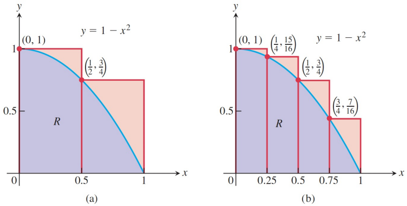
Area and Estimating with Finite Sums

We look at the graph $y = 1 - x^2$ on the interval $[0, 1]$.

First, divide the interval $[0, 1]$ into two equal subintervals $[0, 0.5]$ and $[0.5, 1]$.

One each subinterval, we will use the left endpoint to establish the height of a rectangle over the subinterval by finding the value of the function there.

Figure: Approximating Area



Area Under a Curve

The first left endpoint, $x = 0$, gives us a value of $y(0) = 1$. We construct a rectangle with height 1 over the interval $[0, 0.5]$. We what we have done is to assume the function has the constant value 1 on this small interval.

The second left endpoint, $x = 0.5$, gives us a value of $y(0.5) = \frac{3}{4}$. We construct a rectangle with height $\frac{3}{4}$ over the interval $[0.5, 1]$. We what we have done is to assume the function has the constant value $\frac{3}{4}$ on this small interval.

The sum of the areas of the two rectangles is

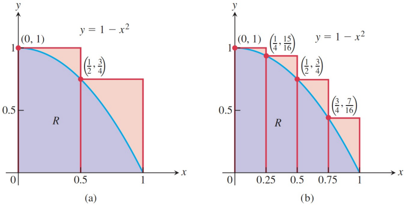
$$\frac{1}{2} \cdot 1 + \frac{1}{2} \cdot \frac{3}{4} = \frac{7}{8}.$$

This is an approximation for the area under the curve $y = 1 - x^2$ over the interval $[0, 1]$.

Area Under a Curve

To get a better estimate of the area, we divide the interval $[0, 1]$ into four subintervals: $[0, 1/4]$, $[1/4, 1/2]$,$[1/2, 3/4]$, and $[3/4, 1]$.

Figure: Approximating Area



Area Under a Curve

The first left endpoint, $x = 0$, gives us a value of $y(0) = 1$. We construct a rectangle with height 1 over the interval $[0, 1/4]$. We what we have done is to assume the function has the constant value 1 on this small interval.

The second left endpoint, $x = 1/4$, gives us a value of $y(1/4) = \frac{15}{16}$. We construct a rectangle with height $\frac{15}{16}$ over the interval $[1/4, 1/2]$. We what we have done is to assume the function has the constant value $\frac{15}{16}$ on this small interval.

The third left endpoint, $x = 1/2$, gives us a value of $y(1/2) = \frac{3}{4}$. We construct a rectangle with height 1 over the interval $[1/2, 3/4]$. We what we have done is to assume the function has the constant value $\frac{3}{4}$ on this small interval.

Area Under a Curve

The fourth left endpoint, $x = 3/4$, gives us a value of $y(3/4) = \frac{7}{16}$. We construct a rectangle with height $\frac{3}{4}$ over the interval $[0.5, 1]$. We what we have done is to assume the function has the constant value $\frac{3}{4}$ on this small interval.

The sum of the areas of the four rectangles is

$$\frac{1}{4} \cdot 1 + \frac{1}{4} \cdot \frac{15}{16} + \frac{1}{4} \cdot \frac{3}{4} + \frac{1}{4} \cdot \frac{7}{16} = \frac{25}{32}.$$

This is a better approximation for the area under the curve $y = 1 - x^2$ over the interval $[0, 1]$.

Area Under a Curve

If you use the point in each subinterval on which the function has its maximum on that subinterval, then the sum is called an **upper sum**. This estimate is larger than (or equal to) the area under the curve.

If you use the point in each subinterval on which the function has its minimum on that subinterval, then the sum is called a **lower sum**. This estimate is smaller than (or equal to) the area under the curve.

The area under the curve is somewhere between the lower sum and the upper sum.

Area Under a Curve

You can actually choose any point in each subinterval. Most customarily, one uses the left endpoint, the right endpoint, or the midpoint of the subinterval. You will (in general) get different estimates for the area under the curve.

Area Under a Curve

How do you get better and better estimates? You divide the interval into more and more subintervals.

Figure: Table of Lower Sums, Midpoint Sums, and Upper Sums

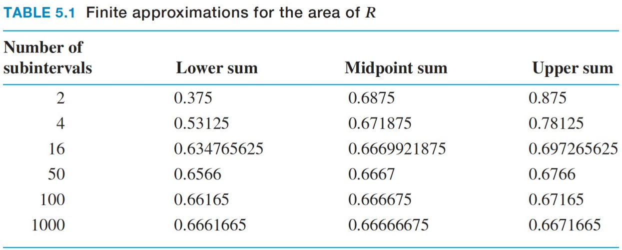


TABLE 5.1 Finite approximations for the area of R

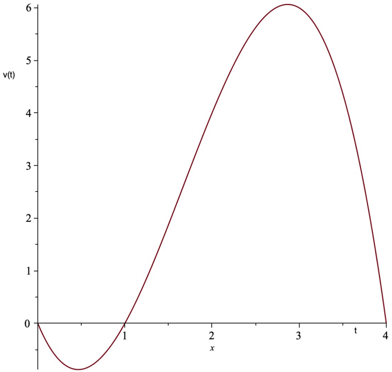
Number of subintervals	Lower sum	Midpoint sum	Upper sum
2	0.375	0.6875	0.875
4	0.53125	0.671875	0.78125
16	0.634765625	0.6669921875	0.697265625
50	0.6566	0.6667	0.6766
100	0.66165	0.666675	0.67165
1000	0.6661665	0.66666675	0.6671665

In particular, it appears the area under the curve $y = 1 - x^2$ over the interval $[0, 1]$ is $2/3$.

Distance Traveled

Suppose we're given the velocity function $v(t)$ for a particle moving along an axis. If we graph this function, we get the graph $y = v(t)$.

Figure: Graph $y = v(t)$



Distance Traveled

If we divide the interval $[1, 4]$ into subintervals, the width of each subintervals if Δt, a change in time, and the height of the approximating rectangle is a velocity. The product of time and velocity is distance.

This interprets the area under the graph $y = v(t)$ as the **distance traveled** between $t = 1$ and $t = 4$.

If we divide the interval $[0, 1]$ into subintervals, the width of each subintervals if Δt, a change in time, and the height of the approximating rectangle is a velocity. The product of time and velocity is distance.

This interprets the area above the graph $y = v(t)$ as the **negative** of the distance traveled between $t = 0$ and $t = 1$. The sign tells you that the object is traveling in the opposite direction on the axis.

Displacement versus Distance Traveled

If we add these two numbers together, we get the **displacement** of the object over the interval $[0, 5]$. That is, you get the distance between where the object started and where it ended. This adds distance to the right and subtracts distance to the left

If we subtract these two numbers together, you get the the **distance traveled** by the object over the interval $[0, 5]$. That is, you get the distance the object traveled to the right plus the distance traveled to the left. Here, we just ignore the direction of travel and add the distances together.

The displacement and the distance traveled are generally not the same thing.

Average Value of a Nonnegative Continuous Function

If you have a nonnegative function $f(x)$ on the interval $[a, b]$, then the area under the curve $y = f(x)$ over the interval $[a, b]$ divided by the length of the interval, $b - a$, gives you the **average value** $\overline{y}$ of the function on $[a, b]$.

If you construct a rectangle with base $[a, b]$ and height $\overline{y}$, then the area of that rectangle equals the area under the curve $y = f(x)$ over the interval $[a, b]$.