Test #1
MATH 4513

Directions: *This is the first hour exam for MATH 4513. The test contains ten problems counting various point values for a total of 100 points. The value of each problem is indicated in parentheses with the problem. Some of the problems are conceptual, some are computational, and some are theoretical. You may use a calculator. You may use any theorems which were proved in class or in the textbook provided that you use them correctly. You may also use results which were proved as part of your homework as long as you cite them here. You may also use anything you learned in MATH 2853.*

Good luck.

My signature below indicates that I have read and understand the instructions printed above and I agree to abide by them.

Name (printed):

Problem 1. Let $\mathbb{F}$ be a field and let n be a positive integer ($n \geq 2$). Let V be the vector space of all $n \times n$ matrices over $\mathbb{F}$. Which of the following sets of matrices A in V are subspaces of V? (5 pts each)

(a) all invertible A;

Solution. The matrices $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ are invertible, but their sum is the zero matrix, which is not invertible, so this set is not a subspace of V.

(b) all non-invertible A;

Solution. The matrices $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ are not invertible, but their sum is the identity matrix, which is invertible, so this set is not a subspace of V.

(c) all A such that $AB = BA$, where B is some fixed matrix in V;

Solution. Let B be a fixed matrix in V and let $W = \{A \in V \mid AB = BA\}$. Let X and Y be in W, so that $XB = BX$ and $YB = BY$. Let k be a scalar. Then

$$\begin{aligned} B(X + kY) &= BX + kBY \\ &= XB + kYB \\ &= (X + kY)B. \end{aligned}$$

This shows this set of matrices is a subspace.

(d) all A such that $A^2 = A$.

Solution. Let $A = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$ and let I be the 2×2 identity matrix. Then $A^2 = A$ and $I^2 = I$, but

$$(A + I)^2 = \begin{bmatrix} 1 & 0 \\ 3 & 4 \end{bmatrix},$$

which is not $A + I$, so this set is not a subspace of V.

Problem 2. If $\mathbb{C}$ is the field of complex numbers, which vectors in $\mathbb{C}^3$ are linear combinations of $(1, 0, -1)$, $(0, 1, 1)$, and $(1, 1, 1)$? You must fully justify your answer. (5 pts)

Solution 1. We wish to see which vectors (a, b, c) can be written as a linear combination of the vectors $(1, 0, -1)$, $(0, 1, 1)$, and $(1, 1, 1)$. This means we have to solve the equation

$$z_1 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + z_2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + z_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}.$$

To do this, we form the augmented coefficient matrix

$$\begin{bmatrix} 1 & 0 & 1 & | & a \\ 0 & 1 & 1 & | & b \\ -1 & 1 & 1 & | & c \end{bmatrix}.$$

Performing Gaussian elimination, we get

$$\begin{bmatrix} 1 & 0 & 1 & | & a \\ 0 & 1 & 1 & | & b \\ 0 & 0 & 1 & | & a+c-b \end{bmatrix}.$$

This matrix has rank 3, so there is a solution for every vector (a, b, c). So, every vector in $\mathbb{C}^3$ is a linear combination of these three vectors.s

Solution 2. Let M be the matrix

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix}.$$

Putting this matrix into reduced row echelon form, we get

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

This matrix has rank 3, so the three columns are linearly independent. Since $\mathbb{C}^3$ has dimension 3, these three columns must form a basis for $\mathbb{C}^3$. So, these vectors must also span $\mathbb{C}^3$. So, every vector in $\mathbb{C}^3$ can be written as a linear combination of these three vectors.

Problem 3. Is the vector $(3, -1, 0, -1)$ in the subspace of $\mathbb{R}^4$ spanned by the vectors $(2, -1, 3, 2)$, $(-1, 1, 1, -3)$, and $(1, 1, 9, -5)$? (5 pts)

Solution 1. The vector $(3, -1, 0, -1)$ lies in the span of $(2, -1, 3, 2)$, $(-1, 1, 1, -3)$, and $(1, 1, 9, -5)$ if and only if the rank of the matrix

$$\begin{bmatrix} 2 & -1 & 1 & 3 \\ -1 & 1 & 1 & -1 \\ 3 & 1 & 9 & 0 \\ 2 & -3 & -5 & -1 \end{bmatrix}$$

has rank at most 3 and the first three columns span the column space. Performing Gaussian elimination on this matrix, we get

$$\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

This says the fourth column is linearly independent of the first three, so the answer is no, the vector $(3, -1, 0, -1)$ is not in the subspace of $\mathbb{R}^4$ spanned by the vectors $(2, -1, 3, 2)$, $(-1, 1, 1, -3)$, and $(1, 1, 9, -5)$.

Solution 2. We determine which vectors (a, b, c, d) lie in the span of $(2, -1, 3, 2)$, $(-1, 1, 1, -3)$, and $(1, 1, 9, -5)$. Taking the augmented coefficient matrix

$$\begin{bmatrix} 2 & -1 & 1 & a \\ -1 & 1 & 1 & b \\ 3 & 1 & 9 & c \\ 2 & -3 & -5 & d \end{bmatrix}$$

and putting it in row echelon form, we get

$$\begin{bmatrix} 1 & -1 & -1 & -b \\ 0 & 1 & 3 & a + 2b \\ 0 & 0 & 0 & c - 5b - 4a \\ 0 & 0 & 0 & d + 4b + a \end{bmatrix}$$

So, in order for this to have a solution, we must have $d + 4b + a = 0$ and $c - 5b - 4a = 0$. The vector $(3, -1, 0, -1)$ doesn't satisfy either of these equations, so it is not a linear combination of the three vectors.

Problem 4. Let W be the set of all $(x_1, x_2, x_3, x_4, x_5)$ in $\mathbb{R}^5$ which satisfy

$$\begin{aligned} 2x_1 - x_2 + \tfrac{4}{3}x_3 - x_4 \qquad &= 0 \\ x_1 \qquad + \tfrac{2}{3}x_3 \qquad - x_5 &= 0 \\ 9x_1 - 3x_2 + 6x_3 - 3x_4 - 3x_5 &= 0 \end{aligned}$$

Find a finite set of vectors which spans W. (5 pts)

Solution. This system has augmented coefficient matrix

$$\begin{bmatrix} 2 & -1 & \frac{4}{3} & -1 & 0 & 0 \\ 1 & 0 & \frac{2}{3} & 0 & -1 & 0 \\ 9 & -3 & 6 & -3 & -3 & 0 \end{bmatrix}.$$

Using Gauss-Jordan Elimination, the reduced row echelon form of this matrix is

$$\begin{bmatrix} 1 & 0 & \frac{2}{3} & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

This matrix corresponds to the system

$$\begin{aligned} x_1 \qquad + \tfrac{2}{3}x_3 \qquad - x_5 &= 0 \\ x_2 \qquad + x_4 - 2x_5 &= 0 \end{aligned}$$

Solving for the leading variables in terms of the free variables, you get

$$\begin{aligned} x_1 &= -\frac{2}{3}x_3 + x_5 \\ x_2 &= -x_4 + 2x_5. \end{aligned}$$

So, the solution space is

$$\begin{bmatrix} -\frac{2}{3}x_3 + x_5 \\ -x_4 + 2x_5 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_3 \begin{bmatrix} -\frac{2}{3} \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

The three vectors on the right form a basis for W.

Problem 5. Let V be the vector space of all 2×2 matrices over the field $\mathbb{F}$. Prove that V has dimension 4 by exhibiting a basis for V which has four elements. Make sure you show your basis is, in fact, a basis by showing the elements are linearly independent and span V. (5 pts)

Solution. *Proof.* Let V be the vector space of all 2×2 matrices over the field $\mathbb{F}$. Let $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be an arbitrary element in V. Then

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

From this expression, it's clear that the set

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

spans V and the elements are linearly independent, so this set is a basis for V. So, V has dimension 4. □

Problem 6. Find the coordinate vector of the vector $(1, 0, 1)$ in the basis of $\mathbb{C}^3$ consisting of the vectors $(2i, 1, 0)$, $(2, -1, 1)$, $(0, 1+i, 1-i)$, in that order. (5 pts)

Solution 1. We need to solve

$$x_1 \begin{bmatrix} 2i \\ 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 1+i \\ 1-i \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

Let M be the matrix with columns $(2i, 1, 0)$, $(2, -1, 1)$, $(0, 1+i, 1-i)$, in that order. Then the last equation can be written

$$M \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

So, the solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = M^{-1} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} - \frac{1}{2}i & -i & -1 \\ -\frac{1}{2}i & -1 & i \\ -\frac{1}{4} + \frac{1}{4}i & \frac{1}{2} + \frac{1}{2}i & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} - \frac{1}{2}i \\ \frac{1}{2}i \\ \frac{3}{4} + \frac{1}{4}i \end{bmatrix}.$$

Solution 2. We form the augmented coefficient matrix:

$$\begin{bmatrix} 2i & 2 & 0 & 1 \\ 1 & -1 & 1+i & 0 \\ 0 & 1 & 1-i & 1 \end{bmatrix}$$

Performing Gauss-Jordan elimination, we get

$$\begin{bmatrix} 1 & 0 & 0 & -\frac{1}{2} - \frac{1}{2}i \\ 0 & 1 & 0 & \frac{1}{2}i \\ 0 & 0 & 1 & \frac{3}{4} + \frac{1}{4}i \end{bmatrix}$$

From the solution is $\begin{bmatrix} -\frac{1}{2} - \frac{1}{2}i \\ \frac{1}{2}i \\ \frac{3}{4} + \frac{1}{4}i \end{bmatrix}$.

Problem 7. Show that the vectors

$$\alpha_1 = (1,1,0,0), \quad \alpha_2 = (0,0,1,1)$$
$$\alpha_3 = (1,0,0,4), \quad \alpha_4 = (0,0,0,2)$$

form a basis for $\mathbb{R}^4$. Find the coordinates of each of the standard basis vectors in the ordered basis $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$. (10 pts)

Solution. We solve

$$x_1\alpha_1 + x_2\alpha_2 + x_3\alpha_3 + x_4\alpha_4 = e_1.$$

Let M be the matrix with columns α_1, α_2, α_3, α_4. This can be written

$$M \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = e_1.$$

Solving this for the coordinate vector, we get

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = M^{-1}e_1,$$

so that the coordinate vector of e_1 with respect to the α-basis is the first column of M^{-1}. Similarly, the coordinate vectors of the other standard basis vectors with respect to α-basis are the other columns of M^{-1}. So, computing M^{-1} using any method, we have

$$[e_1]_\alpha = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -2 \end{bmatrix}, \quad [e_2]_\alpha = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 2 \end{bmatrix}, \quad [e_3]_\alpha = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1/2 \end{bmatrix}, \quad [e_4]_\alpha = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1/2 \end{bmatrix}.$$

Problem 8. Let V be the complex vector space of 2×2 matrices with complex entries. Let

$$A = \begin{bmatrix} 1 & -1 \\ -4 & 4 \end{bmatrix}.$$

and let T be the linear transformation on V defined by $T(X) = AX$. What is the rank of T? Can you describe T^2? (10 pts)

Solution. A basis for V is

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

as in a previous problem. We compute the image of these four matrices under T:

$$T\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ -4 & 0 \end{bmatrix}, \tag{1}$$

$$T\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 1 \\ 0 & -4 \end{bmatrix}, \tag{2}$$

$$T\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right) = \begin{bmatrix} -1 & 0 \\ 4 & 0 \end{bmatrix}, \tag{3}$$

$$T\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} 0 & -1 \\ 0 & 4 \end{bmatrix}. \tag{4}$$

Matrices (1) and (3) are clearly dependent as are matrices (2) and (4). However, matrices (1) and (2) are independent and span the image. So the rank of this transformation is 2.

The linear transformation T^2 is given by the matrix A^2. Computing this, we get

$$A^2 = \begin{bmatrix} 5 & -5 \\ -20 & 20 \end{bmatrix} = 5 \begin{bmatrix} 1 & -1 \\ -4 & 4 \end{bmatrix} = 5A.$$

So, it looks to me like A^2 simply does what A does and then multiplies each vector by 5.

Problem 9. Let W_1 and W_2 be subspaces of a vector space V such that $W_1 + W_2 = V$ and $W_1 \cap W_2 = \{0\}$. Prove that for each vector α in V, there are *unique* vectors $\alpha_1 \in W_1$ and $\alpha_2 \in W_2$ such that $\alpha = \alpha_1 + \alpha_2$. (10 pts)

Solution. *Proof.* Let W_1 and W_2 be subspaces of a vector space V such that $W_1+W_2 = V$ and $W_1 \cap W_2 = \{0\}$. Let $\alpha \in V$. Since $W_1 + W_2 = V$, there exist $\alpha_1 \in W_1$ and $\alpha_2 \in W_2$ so that $\alpha = \alpha_1 + \alpha_2$.

Suppose there exist $\alpha_1' \in W_1$ and $\alpha_2' \in W_2$ so that $\alpha = \alpha_1' + \alpha_2'$ also. Then $\alpha_1 + \alpha_2 = \alpha_1' + \alpha_2'$ and so $\alpha_1 - \alpha_1' = \alpha_2 - \alpha_2'$, and this lies in $W_1 \cap W_2 = \{0\}$. Hence $\alpha_1 - \alpha_1' = \alpha_2 - \alpha_2' = 0$, so that $\alpha_1 = \alpha_1'$ and $\alpha_2 = \alpha_2'$.

So, there exist unique vectors $\alpha_1 \in W_1$ and $\alpha_2 \in W_2$ such that $\alpha = \alpha_1 + \alpha_2$. □

Problem 10. Let V be the vector space of all functions from $\mathbb{R}$ into $\mathbb{R}$; let V_e, be the subset of even functions, $f(-x) = f(x)$; let V_o, be the subset of odd functions, $f(-x) = -f(x)$. (5 pts each)

(a) Prove that V_e and V_o are subspaces of V.

Solution. *Proof.* Let f, $g \in V_e$ and let $k \in \mathbb{R}$. Then

$$(f + kg)(-x) = f(-x) + kg(-x) = f(x) + kg(x) = (f + kg)(x),$$

so V_e is a subspace. The proof that V_o is a subspace is analogous. □

(b) Prove that $V_e + V_o = V$.

Solution. *Proof.* Let $f \in V$. Let

$$g(x) = \frac{1}{2}(f(x) + f(-x)] \text{ and } h(x) = \frac{1}{2}(f(x) - f(-x)].$$

Then $g \in V_e$, $h \in V_o$, and $f = g + h$. This proves $V_e + V_o = V$. $\square$

(c) Prove that $V_e \cap V_o = \{0\}$.

Solution. *Proof.* Let $f \in V_e \cap V_o$. Then for all $x \in \mathbb{R}$,

$$f(-x) = -f(x) \text{ and } f(-x) = f(x)$$

This shows $f(x) = -f(x)$ for all $x \in \mathbb{R}$, so $f \equiv 0$. Hence $V_e \cap V_o = \{0\}$.

□