Problem Set #9 Solutions
Due Thursday, October 16

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Problem 4.1.1. Label the following statements as true or false.

(a) The function $\det : M_{2\times 2}(\mathbb{F}) \to \mathbb{F}$ is a linear transformation.

(b) The determinant of a 2×2 matrix is a linear function of each row of the matrix when the other row is held fixed.

(c) If $A \in M_{2\times 2}(\mathbb{F})$ and $\det(A) = 0$, then A is invertible.

(d) If u and v are vectors in $\mathbb{R}^2$ emanating from the origin, then the area of the parallelogram having u and v as adjacent sides is

$$\det \begin{pmatrix} u \\ v \end{pmatrix}$$

(e) A coordinate system is right-handed if and only if its orientation equals 1.

Solution. (a) False. If $M = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $N = -M$, then $\det(M) = \det(N) = 1$, but $M + N$ is the zero matrix, so its determinant is zero.

(b) True. This is one of the basic properties of determinants.

(c) False. The correct statement is:

If $A \in M_{2\times 2}(\mathbb{F})$ and $\det(A) \neq 0$, then A is invertible.

(d) False. The value

$$\det \begin{pmatrix} u \\ v \end{pmatrix}$$

is the **signed** area of the parallelogram.

(e) True. This is Exercise 4.1.12.

Problem 4.1.5. Prove that if B is the matrix obtained by interchanging the rows of a 2×2 matrix A, then $\det(B) = -\det(A)$.

Solution. *Proof.* Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Let $B = \begin{pmatrix} c & d \\ a & b \end{pmatrix}$.

Then

$$\det(B) = bc - ad = -(ad - bc) = -\det(A)\,.$$

□

Problem 4.1.6. Prove that if the two columns of $A \in M_{2\times 2}(\mathbb{F})$ are identical, then $\det(A) = 0$.

Solution. *Proof.* Let $A = \begin{pmatrix} a & a \\ c & c \end{pmatrix}$. Then

$$\det(A) = ac - ac = 0.$$

□

Problem 4.1.7. Prove that $\det\left(A^t\right) = \det(A)$ for any $A \in M_{2\times 2}(\mathbb{F})$.

Solution. *Proof.* Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then $A^t = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$. Computing, we see that $\det(A) = ad - bc = \det\left(A^t\right)$.

$\square$

Problem 4.1.10. The **classical adjoint** of a 2×2 matrix $A \in M_{2\times 2}(\mathbb{F})$ is the matrix

$$C = \begin{pmatrix} A_{22} & -A_{12} \\ -A_{21} & A_{11} \end{pmatrix}$$

Prove that

(a) $CA = AC = [\det(A)]I$.

(b) $\det(C) = \det(A)$.

(c) The classical adjoint of A^t is C^t.

(d) If A is invertible, then $A^{-1} = [\det(A)]^{-1}C$.

Solution. (a) $CA = AC = [\det(A)]I$.

Proof. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Computing, we see that $C = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$. Then

$$\begin{aligned} AC &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \\ &= \begin{pmatrix} ad - bc & 0 \\ 0 & ad - bc \end{pmatrix} \\ &= (ad - bc) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= [\det(A)]I. \end{aligned}$$

and

$$\begin{aligned} CA &= \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \\ &= \begin{pmatrix} ad - bc & 0 \\ 0 & ad - bc \end{pmatrix} \\ &= (ad - bc) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= [\det(A)]I. \end{aligned}$$

□

(b) $\det(C) = \det(A)$.

Proof. Using the notation in part (a), we have

$$\det(C) = \det \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = ad - bc = \det(A)\,.$$

□

(c) The classical adjoint of A^t is C^t.

Proof. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Computing, we see that $C = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$. We also have $A^t = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$. Computing, we see that the classical adjoint of A^t is $\begin{pmatrix} d & -c \\ -b & a \end{pmatrix} = C^t$.

□

(d) If A is invertible, then $A^{-1} = [\det(A)]^{-1}C$.

Proof. This follows immediately from (a). □