Problem Set #8 Solutions
Due Thursday October 9

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Problem 3.4.1. Label the following statements as true or false.

(a) If $(A'|b')$ is obtained from $(A|b)$ by a finite sequence of elementary column operations, then the systems $Ax = b$ and $A'x = b'$ are equivalent.

(b) If $(A'|b')$ is obtained from $(A|b)$ by a finite sequence of elementary row operations, then the systems $Ax = b$ and $A'x = b'$ are equivalent.

(c) If A is an $n \times n$ matrix with rank n, then the reduced row echelon form of A is I_n.

(d) Any matrix can be put in reduced row echelon form by means of a finite sequence of elementary row operations.

(e) If $(A|b)$ is in reduced row echelon form, then the system $Ax = b$ is consistent.

(f) Let $Ax = b$ be a system of m linear equations in n unknowns for which the augmented matrix is in reduced row echelon form. If this system is consistent, the the dimension of the solution set of $Ax = 0$ is $n - r$, where r equals the number of nonzero rows in A.

(g) If a matrix A is transformed by elementary row operations into a matrix A' in reduced row echelon form, then the number of nonzero rows in A' equals the rank of A.

Solution. (a) False. This is true for row operations, not column operations.

(b) True. This is true for row operations.

(c) True.

(d) True.

(e) False.

(f) True.

(g) True.

Problem 3.4.2. Use Gaussian elimination to solve the following systems of linear equations.

(b) $$\begin{cases} x_1 - 2x_2 - x_3 = 1 \\ 2x_1 - 3x_2 + x_3 = 6 \\ 3x_1 - 5x_2 = 7 \\ x_1 + 5x_3 = 9 \end{cases}$$

(f) $$\begin{cases} x_1 + 2x_2 - x_3 + 3x_4 = 2 \\ 2x_1 + 4x_2 - x_3 + 6x_4 = 5 \\ x_2 + 2x_4 = 3 \end{cases}$$

Solution. (b) The augmented coefficient matrix for this system is

$$\left(\begin{array}{ccc|c} 1 & -2 & -1 & 1 \\ 2 & -3 & 1 & 6 \\ 3 & -5 & 0 & 7 \\ 1 & 0 & 5 & 9 \end{array}\right)$$

Add -2 times row 1 to row 2, add -3 times row 1 to row 3, and add -1 times row 1 to row 4. We get

$$\left(\begin{array}{ccc|c} 1 & -2 & -1 & 1 \\ 0 & 1 & 3 & 4 \\ 0 & 1 & 3 & 4 \\ 0 & 2 & 6 & 8 \end{array}\right)$$

Add 2 times row 2 to row 1, add -1 times row 2 to row 3, and add -2 times row 2 to row 4. We get

$$\left(\begin{array}{ccc|c} 1 & 0 & 5 & 9 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

We have one free variable x_3. The solution is

$$\begin{aligned} x_1 &= 9 - 5x_3 \\ x_2 &= 4 - 3x_3 \\ x_3 &= x_3 \end{aligned}$$

(f) The augmented coefficient matrix for this system is

$$\left(\begin{array}{cccc|c} 1 & 2 & -1 & 3 & 2 \\ 2 & 4 & -1 & 6 & 5 \\ 0 & 1 & 0 & 2 & 3 \end{array}\right)$$

Add -2 times row 1 to row 2. We get

$$\left(\begin{array}{cccc|c} 1 & 2 & -1 & 3 & 2 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 2 & 3 \end{array}\right)$$

Swap row 2 and row 3. We get

$$\left(\begin{array}{cccc|c} 1 & 2 & -1 & 3 & 2 \\ 0 & 1 & 0 & 2 & 3 \\ 0 & 0 & 1 & 0 & 1 \end{array}\right)$$

Add -2 times row 2 to row 1. We get

$$\left(\begin{array}{cccc|c} 1 & 0 & -1 & -1 & -4 \\ 0 & 1 & 0 & 2 & 3 \\ 0 & 0 & 1 & 0 & 1 \end{array}\right)$$

Add 1 times row 3 to row 1. We get

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & -1 & -3 \\ 0 & 1 & 0 & 2 & 3 \\ 0 & 0 & 1 & 0 & 1 \end{array}\right)$$

We have one free variable x_4. The solution is

$$\begin{aligned} x_1 &= -3 + x_4 \\ x_2 &= 3 - 2x_4 \\ x_3 &= 1 \\ x_4 &= x_4. \end{aligned}$$

Problem 3.4.4. For each of the systems that follow, apply Exercise 3 to determine whether the system is consistent. If the system is consistent, find all solutions. Finally, find a basis for the solution set of the corresponding homogeneous system.

(b)

$$\begin{aligned} x_1 + x_2 - 3x_3 + x_4 &= -2 \\ x_1 + x_2 + x_3 - x_4 &= 2 \\ x_1 + x_2 - x_3 &= 0 \end{aligned}$$

Solution. (b) We form the augmented coefficient matrix

$$\left(\begin{array}{cccc|c} 1 & 1 & -3 & 1 & -2 \\ 1 & 1 & 1 & -1 & 2 \\ 1 & 1 & -1 & 0 & 0 \end{array}\right)$$

Performing Gauss-Jordan elimination, we get

$$\left(\begin{array}{cccc|c} 1 & 1 & 0 & -\frac{1}{2} & 1 \\ 0 & 0 & 1 & -\frac{1}{2} & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Since the rank of the coefficient matrix and the rank of the augmented coefficient matrix are equal, there is a solution.

From the reduced row echelon form, we can read off the solution space

$$\left\{ \begin{pmatrix} 1 - x_2 + \frac{1}{2}x_4 \\ x_2 \\ 1 + \frac{1}{2}x_4 \\ x_4 \end{pmatrix} \right\} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \\ 1 \end{pmatrix} \right\}.$$

From this, we can read off the solution space to the associated homogeneous system:

$$\left\{ x_2 \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \\ 1 \end{pmatrix} \right\}.$$

Problem 3.4.5. Let the reduced row echelon form of A be

$$\begin{pmatrix} 1 & 0 & 2 & 0 & -2 \\ 0 & 1 & -5 & 0 & -3 \\ 0 & 0 & 0 & 1 & 6 \end{pmatrix}$$

Determine A if the first, second, and fourth columns of A are

$$\begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}, \text{ and } \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix},$$

respectively.

Solution. To solve this, start with the original matrix A:

$$A = \begin{pmatrix} 1 & 0 & a & 1 & d \\ -1 & -1 & b & -2 & e \\ 3 & 1 & c & 0 & f \end{pmatrix}.$$

We want the reduced row echelon form of A be

$$R = \begin{pmatrix} 1 & 0 & 2 & 0 & -2 \\ 0 & 1 & -5 & 0 & -3 \\ 0 & 0 & 0 & 1 & 6 \end{pmatrix}.$$

Now we perform Gauss-Jordan elimination. For the matrix A, add row 1 to row 2 and add -3 times row 1 to row 3:

$$\begin{pmatrix} 1 & 0 & a & 1 & d \\ 0 & -1 & a+b & -1 & d+e \\ 0 & 1 & c-3a & -3 & f-3d \end{pmatrix}.$$

Next, swap row 2 and row 3 and add row 2 to row 3:

$$\begin{pmatrix} 1 & 0 & a & 1 & d \\ 0 & 1 & c-3a & -3 & f-3d \\ 0 & 0 & b-2a+c & -4 & e-2d+f \end{pmatrix}.$$

Since the $(3,3)$ entry must now be zero, we set $c = 2a - b$ and recompute:

$$\begin{pmatrix} 1 & 0 & a & 1 & d \\ 0 & 1 & -a-b & -3 & f-3d \\ 0 & 0 & 0 & -4 & e-2d+f \end{pmatrix}.$$

Next, multiply row 3 by $-1/4$:

$$\begin{pmatrix} 1 & 0 & a & 1 & d \\ 0 & 1 & -a-b & -3 & f-3d \\ 0 & 0 & 0 & 1 & -\frac{1}{4}e+\frac{1}{2}d-\frac{1}{4}f \end{pmatrix}.$$

Next, add -1 times row 3 to row 1 and 3 times row 3 to row 2:

$$\begin{pmatrix} 1 & 0 & a & 0 & \frac{1}{2}d+\frac{1}{4}e+\frac{1}{4}f \\ 0 & 1 & -a-b & 0 & \frac{1}{4}f-\frac{3}{2}d-\frac{3}{4}e \\ 0 & 0 & 0 & 1 & -\frac{1}{4}e+\frac{1}{2}d-\frac{1}{4}f \end{pmatrix}.$$

Comparing this to the reduced row echelon form given in the problem, we see that

$$\begin{cases} a = 2 \\ -a-b = -5 \end{cases}$$

So, $a = 2$ and $b = 3$.

Now, looking at the last column and comparing it to the reduced row echelon form given in the problem, we see that

$$\begin{cases} \dfrac{1}{2}d+\dfrac{1}{4}e+\dfrac{1}{4}f = -2 \\ \dfrac{1}{4}f-\dfrac{3}{2}d-\dfrac{3}{4}e = -3 \\ -\dfrac{1}{4}e+\dfrac{1}{2}d-\dfrac{1}{4}f = 6. \end{cases}$$

Solving this by your favorite method yields $d = 4$, $e = -7$, and $f = -9$.

So, we see

$$A = \begin{pmatrix} 1 & 0 & 2 & 1 & 4 \\ -1 & -1 & 3 & -2 & -7 \\ 3 & 1 & 1 & 0 & -9 \end{pmatrix}.$$