Problem Set #7 Solutions
Due Tuesday, October 7

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Problem 3.2.3. Prove that for any $m \times n$ matrix A, $\operatorname{rank}(A) = 0$ if and only if A is the zero matrix.

Solution. $(\Leftarrow)$ Certainly the rank of the zero matrix is zero.

$(\Rightarrow)$ Let A be an $m \times n$ matrix with rank zero.

Suppose A is not the zero matrix. Then A has at least one nonzero row and therefore at least one linearly independent row. So, $\operatorname{rank}(A) \geq 1$. This is a contradiction.

Problem 3.2.5. For each of the following matrices, compute the rank and the inverse if it exists.

(h)

$$\begin{pmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & -1 & 2 \\ 2 & 0 & 1 & 0 \\ 0 & -1 & 1 & -3 \end{pmatrix}$$

Solution. (h) This matrix has rank 3. The third row is the sum of the other three rows.

Problem 3.2.6. For each of the following linear transformations T, determine whether T is invertible, and compute T^{-1} if it exists.

(b) $T : P_2(\mathbb{R}) \to P_2(\mathbb{R})$ defined by $T(f(x)) = (x+1)f'(x)$.

(f) $T : M_{2\times 2}(\mathbb{R}) \to \mathbb{R}^4$ defined by

$$T(A) = (\operatorname{tr}(A), \operatorname{tr}(A^t), \operatorname{tr}(EA), \operatorname{tr}(AE)),$$

where

$$E = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Solution. (b) Here T is not invertible since it has rank 2. All constant functions lie in the null space of T.

(f) Let $T : M_{2\times 2}(\mathbb{R}) \to \mathbb{R}^4$ be defined by

$$T(A) = (\operatorname{tr}(A), \operatorname{tr}(A^t), \operatorname{tr}(EA), \operatorname{tr}(AE)),$$

where

$$E = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

We compute

$$\begin{aligned} T\begin{pmatrix} a & b \\ c & d \end{pmatrix} &= \begin{pmatrix} a+d \\ a+d \\ b+c \\ b+c \end{pmatrix} \\ &= a\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + b\begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} + c\begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} + d\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \end{aligned}$$

From this it's easy to see this is not invertible since the rank of the map is 2. The null space is the span of the matrices $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

Problem 3.2.8. Let A be an $m \times n$ matrix. Prove that if c is any nonzero scalar, then $\operatorname{rank}(cA) = \operatorname{rank}(A)$

Solution. Let A be an $m \times n$ matrix and let c be a nonzero scalar. If we set

$$E = \begin{pmatrix} c & 0 & 0 & \cdots & 0 \\ 0 & c & 0 & \cdots & 0 \\ 0 & 0 & c & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & c \end{pmatrix},$$

then

$$cA = EA,$$

Since E is invertible, A and cA have the same rank.

Problem 3.3.4. For each system of linear equations with the invertible coefficient matrix A,

(1) Compute A^{-1}.

(2) Use A^{-1} to solve the system.

(b)

$$\begin{aligned} x_1 + 2x_2 - x_3 &= 5 \\ x_1 + x_2 + x_3 &= 1 \\ 2x_1 - 2x_2 + x_3 &= 4 \end{aligned}$$

Solution. (b)

$$A^{-1} = \begin{pmatrix} \frac{1}{3} & 0 & \frac{1}{3} \\ \frac{1}{9} & \frac{1}{3} & -\frac{2}{9} \\ -\frac{4}{9} & \frac{2}{3} & -\frac{1}{9} \end{pmatrix}.$$

The solution is then

$$A^{-1}\mathbf{b} = \begin{pmatrix} \frac{1}{3} & 0 & \frac{1}{3} \\ \frac{1}{9} & \frac{1}{3} & -\frac{2}{9} \\ -\frac{4}{9} & \frac{2}{3} & -\frac{1}{9} \end{pmatrix} \begin{pmatrix} 5 \\ 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ -2 \end{pmatrix}.$$

Problem 3.3.9. Prove that the system of linear equations $Ax = b$ has a solution if and only if $b \in R(L_A)$.

Solution. *Proof.* The range of L_A is the column space of the matrix A. The equation $Ax = b$ has a solution if and only if b is in that image of L_A. Putting these two together, $Ax = b$ has a solution if and only if $b \in R(L_A)$. $\square$

Problem 3.3.10. Prove or give a counterexample to the following statement: If the coefficient matrix of a system of m linear equations in n unknowns has rank m, then the system has a solution.

Solution.

Proposition. *If the coefficient matrix of a system of m linear equations in n unknowns has rank m, then the system has a solution.*

Proof. Let M be the coefficient matrix for a system of m linear equations in n unknowns and suppose $\operatorname{rank}(M) = m$. Then the associated linear transformation $L_M : \mathbb{R}^n \to \mathbb{R}^m$ has rank m and is therefore surjective. Thus, $R(L_M) = \mathbb{R}^m$ and the system must therefore have a solution. □