Problem Set #6 Solutions
Due Thursday, September 25

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Problem 2.6.1. Label the following statements as true or false. Assume that all vector spaces are finite-dimensional.

(a) Every linear transformation is a linear functional.

(b) A linear functional defined on a field may be represented as a 1×1 matrix.

(c) Every vector space is isomorphic to its dual space.

(d) Every vector space is the dual of some other vector space.

(e) If T is an isomorphism from V into V^* and β is a finite ordered basis for V, then $T(\beta) = \beta^*$.

(f) If T is a linear transformation from V to W, then the domain of $(T^t)^t$ is $(V^*)^*$.

(g) If V is isomorphic to W, then V^* is isomorphic to W^*.

(h) The derivative of a function may be considered as a linear functional on the vector space of differentiable functions.

Solution. (a) False. A linear functional is a linear transformation from V into $\mathbb{F}$.

(b) False. Let $T : \mathbb{C} \to \mathbb{R}$ be a linear functional. Since $\mathbb{C}$ has dimension 2 as a vector space over $\mathbb{R}$, its matrix is a 1×2 matrix.

(c) This is true for finite dimensional vector spaces since V and V^* have the same dimension. For infinite dimensional vector spaces, this is false.

Let V be the real vector space of all sequences (a_n) of real numbers where $a_n = 0$ for all but finite many $n \in \mathbb{N}$. Then V is countable. However, V^* consists of **all** sequences of real numbers, and this set is uncountable.

(d) True. It's true that $(V^*)^* = V$, so every vector space is the dual of some other vector space.

(e) False. An isomorphism $T : V \to V^*$ can be given taking any basis of V to any basis for V^*. It does not have to take β to β^*.

(f) True.

(g) This is true if V and W are finite dimensional. They have the same dimension.

(h) False. Whereas differentiation is a linear map, it does not go into $\mathbb{F}$, so it's not a linear functional.

Problem 2.6.4. Let $V = \mathbb{R}^3$, and define f_1, f_2, $f_3 \in V^*$ as follows:

$$f_1(x, y, z) = x - 2y, \quad f_2(x, y, z) = x + y + z, \quad f_3(x, y, z) = y - 3z.$$

Prove that $\{f_1, f_2, f_3\}$ is a basis for V^*, and then find a basis for V for which it is the dual basis.

Solution. *Proof.* Since V^* has the same dimension as V—three—we need only show f_1, f_2, and f_3 are linearly independent.

Suppose there are scalars a_1, a_2, $a_3 \in \mathbb{R}$ so that $a_1 f_1 + a_2 f_2 + a_3 f_3 = 0$. Then

$$\begin{aligned} 0 &= a_1 f_1 + a_2 f_2 + a_3 f_3 \\ &= a_1(x - 2y) + a_2(x + y + z) + a_3(y - 3z) \\ &= (a_1 + a_2)x + (-2a_1 + a_2 + a_3)y + (a_2 - 3_3)z. \end{aligned}$$

This gives us a linear homogeneous system of three equations in three unknowns:

$$\begin{aligned} a_1 + a_2 &= 0 \\ -2a_1 + a_2 + a_3 &= 0 \\ a_2 - 3_3 &= 0. \end{aligned}$$

Solving this using standard techniques from College Algebra gives the only solution as $a_1 = a_2 = a_3 = 0$. So, f_1, f_2, f_3 are linearly independent. Thus $\{f_1, f_2, f_3\}$ is a basis for V^*. □

To find a basis for V dual to this one, we need to solve

$$\begin{aligned} x - 2y &= 1 \\ x + y + z &= 0 \\ y - 3z &= 0. \end{aligned}$$

This gives $\mathbf{b}_1 = (2/5, -3/10, -1/10)$.

We need to solve

$$\begin{aligned} x - 2y &= 0 \\ x + y + z &= 1 \\ y - 3z &= 0. \end{aligned}$$

This gives $\mathbf{b}_2 = (3/5, 3/10, 1/10)$.

We need to solve

$$\begin{aligned} x - 2y &= 0 \\ x + y + z &= 0 \\ y - 3z &= 1. \end{aligned}$$

This gives $\mathbf{b}_3 = (1/5, 1/10, -3/10)$.

The basis $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3)$ is the dual basis for V.

Problem 2.6.5. Let $V = P_1(\mathbb{R})$, and, for $p(x) \in V$, define $f_1,\ f_2 \in V^*$ by

$$f_1(p(x)) = \int_0^1 p(t)\,dt \text{ and } f_2(p(x)) = \int_0^2 p(t)\,dt.$$

Prove that $\{f_1, f_2\}$ is a basis for V^*, and find a basis for V for which it is the dual basis.

Solution. We know that $\dim(A^*) = \dim(V) = 2$, so we only need to show f_1 and f_2 are linearly independent to show they are a basis.

Suppose $c_1 f_1 + c_2 f_2 = 0$. Then

$$\begin{aligned} 0 &= (c_1 f_1 + c_2 f_2)(1) \\ &= c_1 \int_0^1 1\,dt + c_2 \int_0^2 1\,dt \\ &= c_1 + 2c_2 \end{aligned}$$

and

$$\begin{aligned} 0 &= (c_1 f_1 + c_2 f_2)(t) \\ &= c_1 \int_0^1 t\,dt + c_2 \int_0^2 t\,dt \\ &= \frac{1}{2}c_1 + 2c_2 \end{aligned}$$

Solving the system

$$\begin{cases} c_1 + 2c_2 = 0 \\ \frac{1}{2}c_1 + 2c_2 = 0 \end{cases}$$

we see $c_1 = c_2 = 0$, so f_1 and f_2 are linearly independent.

To find a dual basis, we must find polynomials $p_1 = a + bt$ and $p_2 = c + dt$, so that $f_i(p_j) = \delta_{ij}$ for $1 \leq i, j \leq 2$.

We compute

$$\begin{aligned} f_1(p_1(t)) &= \int_0^1 a + bt\,dt = a + \frac{1}{2}b \\ f_1(p_2(t)) &= \int_0^1 c + dt\,dt = c + \frac{1}{2}d \\ f_2(p_1(t)) &= \int_0^2 a + bt\,dt = 2a + 2b \\ f_2(p_2(t)) &= \int_0^2 c + dt\,dt = 2c + 2d. \end{aligned}$$

We solve the system of equations

$$a + \frac{1}{2}b = 1$$
$$c + \frac{1}{2}d = 0$$
$$2a + 2b = 0$$
$$2c + 2d = 1$$

Solving, we get $a = 2$, $b = -2$, $c = -\frac{1}{2}$, and $d = 1$. So, the dual basis is

$$p_1(x) = 2 - 2x$$
$$p_2(x) = -\frac{1}{2} + x.$$

Problem 2.6.8. Show that every plane through the origin in $\mathbb{R}^3$ may be identified with the null space of a vector in $(\mathbb{R}^3)^*$. State an analogous result for $\mathbb{R}^2$.

Solution. Let Π be a plane through the origin in $\mathbb{R}^3$. Then Π has an equation of the form $a_1x + a_2y + a_3z = 0$. Define $L_\Pi : \mathbb{R}^3 \to \mathbb{R}$ by $L_\Pi(x_1, x_2, x_3) = a_1x_1 + a_2x_2 + a_3x_3$. It's easy to check that L_Π is linear, so $L_\Pi \in (\mathbb{R}^3)^*$ and $N(L_\Pi) = \Pi$.

Problem 3.1.1. Label the following statements as true or false.

(a) An elementary matrix is always square.

(b) The only entries of an elementary matrix are zeros and ones.

(c) The $n \times n$ identity matrix is an elementary matrix.

(d) The product of two $n \times n$ elementary matrices is an elementary matrix.

(e) The inverse of an elementary matrix is an elementary matrix.

(f) The sum of two $n \times n$ elementary matrices is an elementary matrix.

(g) The transpose of an elementary matrix is an elementary matrix.

(h) If B is a matrix that can be obtained by performing an elementary row operation on a matrix A, then B can also be obtained by performing an elementary column operation on A.

(i) If B is a matrix that can be obtained by performing an elementary row operation on a matrix A, then A can be obtained by performing an elementary row operation on B.

Solution. (a) True. An elementary matrix is defined to be an $n \times n$ matrix.

(b) False. The matrix $\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$ is an elementary matrix that adds twice row one to row two.

(c) True. If you start with I_n (for $n \geq 2$) and add zero times row one to row two, you get I_n. This elementary matrix does nothing.

(d) False. An elementary matrix is a matrix obtained from I_n by a single elementary row operation. The product of two of these performs two elementary row operations. In general, this will not be the result of a single elementary row operation.

(e) True. The inverse operation of an elementary row operation is another elementary row operation, so the inverse of an elementary matrix is another elementary matrix.

(f) False. The matrices $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ are both elementary matrices, but their sum $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ is not—it's not invertible and every elementary matrix is invertible.

(g) True.

(h) False. Let $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$. If we add the first row of A to the second row of A, we get B. However, any elementary column operation on A will leave the second row all zero, so you cannot get B.

(i) True.

Problem 3.1.2. Let

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ 1 & -1 & 1 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 & 3 \\ 1 & -2 & 1 \\ 1 & -3 & 1 \end{pmatrix}, C = \begin{pmatrix} 1 & 0 & 3 \\ 0 & -2 & -2 \\ 1 & -3 & 1 \end{pmatrix}.$$

Find an elementary operation that transforms A into B and an elementary operation that transforms B into C. By means of several additional operations, transform C into I_3.

Solution. Start with A. If we add -2 times column 1 to column 2, you get B.

Start with B. If we add -1 times row 1 to row 2, you get C.

Start with C:

$$\begin{pmatrix} 1 & 0 & 3 \\ 0 & -2 & -2 \\ 1 & -3 & 1 \end{pmatrix}$$

Add -1 times row 1 to row 3:

$$\begin{pmatrix} 1 & 0 & 3 \\ 0 & -2 & -2 \\ 0 & -3 & -2 \end{pmatrix}$$

Multiply row 2 by $-1/2$:

$$\begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & -3 & -2 \end{pmatrix}$$

Add 3 times row 2 to row 3:

$$\begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

Now, add -3 times row 3 to row 1 and add -1 times row 3 to row 2. This gives the identity matrix.

Problem 3.1.3. Use the proof of Theorem 3.2 to obtain the inverse of each of the following elementary matrices.

(a) $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

(b) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

(c) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}$

Solution. (a) This is the elementary matrix the interchanges row 1 and row 3. This elementary row operation is its own inverse, so this matrix is its own inverse.

(b) This is the elementary matrix of the elementary operation that multiplies row 2 by 3. The inverse operation multiplies row 2 by $\frac{1}{3}$. So the inverse matrix is $\begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

(c) This is the elementary matrix of the elementary operation that adds -2 times row 1 to row 3. The inverse operation adds 2 times row 1 to row 3. So the inverse matrix is $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}$.

Problem 3.1.5. Prove that E is an elementary matrix if and only if E^t is.

Solution. We consider the three types of elementary matrix operations.

The matrix E is obtained from I_n by swapping row i and row j if and only if E^t is obtained from I_n by swapping column i and column j.

The matrix E is obtained from I_n by multiplying row i by λ if and only if E^t is obtained from I_n by by multiplying column i by λ.

The matrix E is obtained from I_n by adding k times row i to row j if and only if E^t is obtained from I_n by adding k times column i to column j.

So, we see E is an elementary matrix if and only if E^t is.