Problem Set #5 Solutions
Due Thursday, September 18

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Problem 2.4.4. Let A and B be $n \times n$ invertible matrices. Prove that AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$.

Solution. *Proof.* Let A and B be $n \times n$ invertible matrices.
Then

$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AA^{-1} = I$$
$$(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}B = I.$$

So, AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$. □

Problem 2.4.5. Let A be invertible. Prove that A^t is invertible and $(A^t)^{-1} = (A^{-1})^t$.

Solution. *Proof.* Let A be an invertible matrix.

Then

$$\begin{aligned} A^t(A^{-1})^t &= (A^{-1}A)^t = I^t = I \\ (A^{-1})^t A^t &= (AA^{-1})^t = I^t = I \end{aligned}$$

So, A^t is invertible and $(A^t)^{-1} = (A^{-1})^t$. □

Problem 2.4.9. Let A and B be $n \times n$ matrices such that AB is invertible. Prove that A and B are invertible. Give an example to show that arbitrary matrices A and B need not be invertible if AB is invertible.

Solution. *Proof.* Let A and B be $n \times n$ matrices such that AB is invertible.

Let M be the inverse matrix of AB.

Let $v \in \mathbb{F}^n$ and suppose $Bv = 0$. Then

$$0 = MA(0) = MA(Bv) = M(AB)v = v,$$

so we see that B has nullity zero and rank n. This says B is invertible.

Now, suppose $v \in \mathbb{F}^n$ and suppose $Av = 0$. Since B has rank n, there exists $w \in \mathbb{F}^n$ so that $B(w) = v$.

$$0 = Av = A(B(w)) = (AB)(w).$$

Since AB is invertible, this implies that $w = 0$, so that $v = B(w) = 0$. It follows that A has nullity zero and rank n. This says A is invertible.

Let

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 1 & 0 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & 1 \\ 2 & 1 \\ 0 & 0 \end{pmatrix}.$$

□

Then

$$AB = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix},$$

which is invertible, but neither A nor B is invertible (since neither is a square matrix).

Problem 2.4.10. Let A and B be $n \times n$ matrices such that $AB = I_n$.

(a) Use Exercise 9 to conclude that A and B are invertible.

(b) Prove $A = B^{-1}$ (and hence $B = A^{-1}$). (We are, in effect, saying that for square matrices, a "one-sided" inverse is a "two-sided" inverse.)

(c) State and prove analogous results for linear transformations defined on finite-dimensional vector spaces.

Solution. Let A and B be $n \times n$ matrices such that $AB = I_n$.

(a) *Proof.* By Exercise 9, since $I_n = AB$ is invertible, A and B are invertible. □

(b) *Proof.* Since $AB = I_n$, we know from (a) that B is invertible.

$$A = AI_n = A(BB^{-1}) = (AB)B^{-1} = I_nB^{-1} = B^{-1}.$$

□

(c)

Proposition. *Let V and W be finite dimensional vector spaces of the same dimension and let $T : V \to W$, $U : W \to V$ be linear transformations satisfying $TU = I_W$, where $I_W : W \to W$ is the identity map. Then T and U are invertible and $T = U^{-1}$.*

Proof. Let V and W be finite dimensional vector spaces of the same dimension and let $T : V \to W$, $U : W \to V$ be linear transformations from satisfying $TU = I_W$, where $I_W : W \to W$ is the identity map. Let $w \in W$ and suppose $U(w) = 0$. Then

$$0 = T(0) = T(U(w)) = (TU)(w) = I_W(w) = w.$$

From this we see that U is injective. Since $\dim(V) = \dim(W)$ and U is injective, it follows from the Dimension Theorem (Theorem 2.3) that U is surjective as well. Hence U has an inverse U^{-1}. Then

$$U^{-1} = I_WU^{-1} = (TU)U^{-1} = T(UU^{-1}) = TI = T.$$

So, $T = U^{-1}$. □

Problem 2.5.2. For each of the following pairs of ordered bases β and β' for $\mathbb{R}^2$, find the change of coordinate matrix that changes β'-coordinates into β-coordinates.

(a) $\beta = \{e_1, e_2\}$ and $\beta' = \{(a_1, a_2), (b_1, b_2)\}$

(d) $\beta = \{(-4, 3), (2, -1)\}$ and $\beta' = \{(2, 1), (-4, 1)\}$

Solution. (a) Let Q change of coordinate matrix that changes β'-coordinates into β-coordinates. The jth column of Q is the image of $Q[\beta'(j)]$ written in terms of β-coordinates. However, this is exactly how β' is given here in the problem. So,

$$Q = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}$$

(d) Let Q change of coordinate matrix that changes β'-coordinates into β-coordinates. The jth column of Q is the image of $Q[\beta'(j)]$ written in terms of β-coordinates. So, we must write $(2, 1)$ and $(-4, 1)$ in terms of the basis β.

We need to solve

$$\begin{pmatrix} -4 & 2 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

We solve this to get $\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$.

Similarly, we need to solve

$$\begin{pmatrix} -4 & 2 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -4 \\ 1 \end{pmatrix}$$

We solve this to get $\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -1 \\ -4 \end{pmatrix}$. So,

$$Q = \begin{pmatrix} 2 & -1 \\ 5 & -4 \end{pmatrix}.$$

Problem 2.5.3. For each of the following pairs of ordered bases β and β' for $P_2(\mathbb{R})$, find the change of coordinate matrix that changes β'-coordinates into β-coordinates.

(a) $\beta = \{x^2, x, 1\}$ and $\beta' = \{a_2x^2 + a_1x + a_0, b_2x^2 + b_1x + b_0, c_2x^2 + c_1x + c_0\}$

Solution. (a) Once again, we have to write each vector in the β'-basis as coordinate vectors with respect to the β-basis. But we can read this off immediately. The change of basis matrix is

$$\begin{pmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_0 & b_0 & c_0 \end{pmatrix}.$$

Problem 2.5.7. In $\mathbb{R}^2$, let L be the line $y = mx$, where $m \neq 0$. Find an expression for $T(x, y)$, where

(a) T is the reflection of $\mathbb{R}^2$ about L.

(b) T is the projection on L along the line perpendicular to L. (See the definition of projection in the exercises of Section 2.1.)

Solution. (a) Let β be the standard basis for $\mathbb{R}^2$ and let β' be the basis $\{(1, m), (-m, 1)\}$. Then we have

$$[I]_{\beta'}^{\beta} = \begin{pmatrix} 1 & -m \\ m & 1 \end{pmatrix}.$$

With respect to the basis β', the reflection has matrix

$$[R]_{\beta'} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The reflection matrix with respect to the standard basis is then

$$\begin{aligned} [R]_{\beta} &= [I]_{\beta'}^{\beta} [R]_{\beta'} [I]_{\beta}^{\beta'} \\ &= [I]_{\beta'}^{\beta} [R]_{\beta'} \left([I]_{\beta'}^{\beta} \right)^{-1} \\ &= \begin{pmatrix} 1 & -m \\ m & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \frac{1}{1+m^2} & \frac{m}{1+m^2} \\ -\frac{m}{1+m^2} & \frac{1}{1+m^2} \end{pmatrix} \\ &= \begin{pmatrix} \frac{1-m^2}{1+m^2} & \frac{2m}{1+m^2} \\ \frac{2m}{1+m^2} & \frac{m^2-1}{1+m^2} \end{pmatrix}. \end{aligned}$$

(b) Let β be the standard basis for $\mathbb{R}^2$ and let β' be the basis $\{(1, m), (-m, 1)\}$. Then we have

$$[I]_{\beta'}^{\beta} = \begin{pmatrix} 1 & -m \\ m & 1 \end{pmatrix}.$$

With respect to the basis β', the projection has matrix

$$[P]_{\beta'} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

The reflection matrix with respect to the standard basis is then

$$
\begin{aligned}
[P]_\beta &= [I]_{\beta'}^{\beta}[P]_{\beta'}[I]_{\beta}^{\beta'} \\
&= [I]_{\beta'}^{\beta}[P]_{\beta'}\left([I]_{\beta'}^{\beta}\right)^{-1} \\
&= \begin{pmatrix} 1 & -m \\ m & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \frac{1}{1+m^2} & \frac{m}{1+m^2} \\ -\frac{m}{1+m^2} & \frac{1}{1+m^2} \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{1+m^2} & \frac{m}{1+m^2} \\ \frac{m}{1+m^2} & \frac{m^2}{1+m^2} \end{pmatrix}
\end{aligned}
$$

Problem 2.5.10. Prove that if A and B are similar $n \times n$ matrices, then $\operatorname{tr}(A) = \operatorname{tr}(B)$.
Hint: Use Exercise 13 of Section 2.3.

Solution. *Proof.* Section 2.3, Exercise 13 says $\operatorname{tr}(AB) = \operatorname{tr}(BA)$.

Suppose A and B are similar matrices. Then there is an invertible matrix Q so that $B = Q^{-1}AQ$. By Exercise 13 of Section 2.3, we have

$$\begin{aligned} \operatorname{tr}(B) &= \operatorname{tr}(Q^{-1}AQ) \\ &= \operatorname{tr}(AQ^{-1}Q) \\ &= \operatorname{tr}(A). \end{aligned}$$

□

Since A and B are arbitrary similar matrices, this shows the traces of any two similar matrices are equal.

Problem 2.5.11. Let V be a finite-dimensional vector space with ordered bases α, β, and γ.

(a) Prove that if Q and R are the change of coordinate matrices that change α-coordinates into β-coordinates and β-coordinates into γ-coordinates, respectively, then RQ is the change of coordinate matrix that changes α-coordinates into γ-coordinates.

(b) Prove that if Q changes α-coordinates into β-coordinates, then Q^{-1} changes β-coordinates into α-coordinates.

Solution. Let V be a finite-dimensional vector space with ordered bases α, β, and γ.

(a) *Proof.* Suppose Q and R are the change of coordinate matrices that change α-coordinates to β-coordinates and β-coordinates to γ-coordinates, respectively. Then for any $\mathbf{v} \in V$, we have

$$[\mathbf{v}]_\beta = Q[\mathbf{v}]_\alpha \quad \text{and} \quad [\mathbf{v}]_\gamma = R[\mathbf{v}]_\beta.$$

Then changes α-coordinates into γ-coordinates.

$$[\mathbf{v}]_\gamma = R[\mathbf{v}]_\beta = R(Q[\mathbf{v}]_\alpha) = (RQ)[\mathbf{v}]_\alpha.$$

Hence, RQ changes α-coordinates into γ-coordinates. □

(b) *Proof.* Suppose Q changes α-coordinates into β-coordinates. Then for any $\mathbf{v} \in V$, we have

$$[\mathbf{v}]_\beta = Q[\mathbf{v}]_\alpha.$$

Then

$$Q^{-1}[\mathbf{v}]_\beta = Q^{-1}(Q[\mathbf{v}]_\alpha) = (Q^{-1}Q)[\mathbf{v}]_\alpha = [\mathbf{v}]_\alpha.$$

Hence, Q^{-1} changes β-coordinates into α-coordinates. □