Problem Set #4 Solutions
Due Thursday, September 11

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Problem 2.2.4. Define

$$T : M_{2\times 2}(\mathbb{R}) \to P_2(\mathbb{R}) \text{ by } T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = (a+b) + (2d)x + bx^2.$$

Let

$$\beta = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}, \text{ and } \gamma = \{1, x, x^2\}.$$

Compute $[T]_\beta^\gamma$.

Solution. For each matrix $M \in \beta$, we have to write $T(M)$ as linear combination of γ.

First,

$$T\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = 1 = 1 + 0x + 0x^2,$$

so the vector $(1, 0, 0, 0)$ goes to the vector $(1, 0, 0)$.

Second,

$$T\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = 1 + x^2 = 1 + 0x + 1x^2,$$

so the vector $(0, 1, 0, 0)$ goes to the vector $(1, 0, 1)$.

Third,

$$T\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = 0 = 0 + 0x + 0x^2,$$

so the vector $(0, 0, 1, 0)$ goes to the vector $(0, 0, 0)$.

Lastly,

$$T\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = 2x = 0 + 2x + 0x^2,$$

so the vector $(0, 0, 0, 1, 0)$ goes to the vector $(0, 2, 0)$.

So,

$$[T]_\beta^\gamma = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

Problem 2.2.8. Let V be an n-dimensional vector space with an ordered basis β. Define $T : V \to \mathbb{F}^n$ by $T(x) = [x]_\beta$. Prove that T is linear.

Solution. Let $\beta = (v_1, v_2, \dots, v_n)$ be an ordered basis for V. Define $T : V \to \mathbb{F}^n$ by $T(x) = [x]_\beta$. That is, if

$$x = a_1 v_1 + a_2 v_2 + \cdots + a_n v_n$$

then $T(x) = [a_1 \ a_2 \ \dots \ a_n]^t$.

Let x, $y \in V$ and let

$$\begin{aligned} x &= a_1 v_1 + a_2 v_2 + \cdots + a_n v_n \\ y &= b_1 v_1 + b_2 v_2 + \cdots + b_n v_n. \end{aligned}$$

Then

$$\begin{aligned} T(x+y) &= [x+y]_\beta \\ &= [a_1 + b_1 \ a_2 + b_2 \ \dots \ a_n + b_n]^t \\ &= [a_1 \ a_2 \ \dots \ a_n]^t + [b_1 \ b_2 \ \dots \ b_n]^t \\ &= T(x) + T(y). \end{aligned}$$

Also, for $\lambda \in \mathbb{F}$, t

$$\begin{aligned} T(\lambda x) &= T(\lambda(a_1 v_1 + a_2 v_2 + \cdots + a_n v_n)) \\ &= T(\lambda a_1 v_1 + \lambda a_2 v_2 + \cdots + \lambda a_n v_n) \\ &= [\lambda a_1 \ \lambda a_2 \ \dots \ \lambda a_n]^t \\ &= \lambda [a_1 \ a_2 \ \dots \ a_n]^t \\ &= \lambda T(x). \end{aligned}$$

So, T is a linear transformation.

Problem 2.2.9. Let V be the vector space of complex numbers over the field $\mathbb{R}$. Define $T : V \to V$ by $T(z) = \overline{z}$, where $\overline{z}$ is the complex conjugate of z. Prove that T is linear, and compute $[T]_\beta$, where $\beta = \{1, i\}$. (Recall by Exercise 38 of Section 2.1 that T is not linear if V is regarded as a vector space over the field $\mathbb{C}$.)

Solution. Let V be the vector space of complex numbers over the field $\mathbb{R}$. Define $T : V \to V$ by $T(z) = \overline{z}$.

Let z, $w \in \mathbb{C}$. Then

$$T(z + w) = \overline{z + w} = \overline{z} + \overline{w} = T(z) + T(w).$$

and, for $\lambda \in \mathbb{R}$,

$$T(\lambda z) = \overline{\lambda z} = \overline{\lambda} \cdot \overline{z} = \lambda \cdot \overline{z} = \lambda T(z).$$

So, T is a linear transformation.

Using $\beta = \{1, i\}$, we see that $T(1) = 1$ and $T(i) = -i$, so

$$T_\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Problem 2.2.15. Let V and W be vector spaces, and let S be a subset of V. Define $S^0 = \{T \in \mathcal{L}(V,W) : T(x) = 0 \text{ for all } x \in S\}$. Prove the following statements.

(a) S_0 is a subspace of $\mathcal{L}(V,W)$.

(b) If S_1 and S_2 are subsets of V and $S_1 \subseteq S_2$, then $S_2^0 \subseteq S_1^0$.

(c) If V_1 and V_2 are subspaces of V, then $(V_1 + V_2)^0 = V_1^0 \cap V_2^0$.

Solution. Let V and W be vector spaces, and let S be a subset of V. Define $S^0 = \{T \in \mathcal{L}(V,W) : T(x) = 0 \text{ for all } x \in S\}$.

(a) *Proof.* Since $T_0(x) = 0$ for all $x \in V$, we have that $T_0(x) = 0$ for all $x \in S$. So, $T_0 \in S^0$.

Suppose $T \in S^0 \in S^0$ and λ is a scalar. Then

$$(\lambda T)(x) = \lambda T(x) = \lambda \cdot 0 = 0,$$

for all $x \in S$. Thus, $\lambda T \in S^0$.

Suppose $T_1, T_2 \in S^0$. Then $T_1(x) = 0$ and $T_2(x) = 0$ for all $x \in S$. But then we have

$$(T_1 + T_2)(x) = T_1(x) + T_2(x) = 0 + 0 = 0$$

for all $x \in S$. Thus, $T_1 + T_2 \in S^0$.

So, S^0 is a subspace of $\mathcal{L}(V,W)$ by Theorem 1.3. □

(b) *Proof.* Let S_1 and S_2 be subsets of V with $S_1 \subseteq S_2$. Let $T \in S_2^0$. Then $T(x) = 0$ for all $x \in S_2$. Since $S_1 \subseteq S_2$, we have $T(x) = 0$ for all $x \in S_1$. Hence, $T \in S_1^0$.

Since $T \in S_2^0$ is arbitrary, $S_2^0 \subseteq S_1^0$.

□

(c) *Proof.* Suppose V_1 and V_2 are subspaces of V.

Since $V_i \subset V_1 + V_2$ for $i = 1, 2$, we have $(V_1 + V_2)^0 \subseteq V_i^0$ for $i = 1, 2$. So, $(V_1 + V_2)^0 \subseteq V_1^0 \cap V_2^0$.

On the other hand, suppose $T \in V_1^0 \cap V_2^0$. Then $T(x) = 0$ for all $x \in V_1$ and all $x \in V_2$. But every element of $V_1 + V_2$ is the sum of elements in V_1 and V_2, it follows that $T(x) = 0$ for all $x \in V_1 + V_2$. That is, $T \in (V_1 + V_2)^0$.

The two inclusions show that $(V_1 + V_2)^0 = V_1^0 \cap V_2^0$. □

Problem 2.3.3. Let $g(x) = 3 + x$. Let $T : P_2(\mathbb{R}) \to P_2(\mathbb{R})$ and $U : P_2(\mathbb{R}) \to \mathbb{R}^3$ be the linear transformations respectively defined by

$$T(f(x)) = f'(x)g(x) + 2f(x) \text{ and } U(a + bx + cx^2) = (a + b, c, a - b).$$

Let β and γ be the standard ordered bases of $P_2(\mathbb{R})$ and $\mathbb{R}^3$, respectively.

(a) Compute $[U]_\beta^\gamma$, $[T]_\beta$, and $[UT]_\beta^\gamma$ directly. Then use Theorem 2.11 to verify your result.

(b) Let $h(x) = 3 - 2x + x^2$. Compute $[h(x)]_\beta$ and $[U(h(x))]_\gamma$. Then use $[U]_\beta^\gamma$ from (a) and Theorem 2.14 to verify your result.

Solution. (a) We compute

$$\begin{aligned} U(1) &= (1, 0, 1) \\ U(x) &= (1, 0, -1) \\ U(x^2) &= (0, 1, 0). \end{aligned}$$

$$[U]_\beta^\gamma = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}.$$

$$\begin{aligned} T(1) &= 0 \cdot (3 + x) + 2 \cdot 1 = 2 \\ T(x) &= 1 \cdot (3 + x) + 2 \cdot x = 3 + 3x \\ T(x^2) &= 2x \cdot (3 + x) + 2x^2 = 6x + 4x^2. \end{aligned}$$

From this we read off

$$[T]_\beta = \begin{pmatrix} 2 & 3 & 0 \\ 0 & 3 & 6 \\ 0 & 0 & 4 \end{pmatrix}.$$

We compute

$$\begin{aligned} (UT)(1) &= U(T(1)) = U(2) = (2, 0, 2) \\ (UT)(x) &= U(T(x)) = U(3 + 3x) = (6, 0, 0) \\ (UT)(x^2) &= U(T(x^2)) = U(6x + 4x^2) = (6, 4, -6). \end{aligned}$$

From this we read off

$$[UT]_\beta^\gamma = \begin{pmatrix} 2 & 6 & 6 \\ 0 & 0 & 4 \\ 2 & 0 & -6 \end{pmatrix}.$$

It's easily checked that

$$[U]_\beta^\gamma [T]_\beta = [UT]_\beta^\gamma$$

(b) We can read off $[h(x)]_\beta = (3, -2, 1)$.

We compute

$$U(h(x)) = U(3 - 2x + x^2) = (1, 1, 5),$$

so that

$$[U(h(x))]_\gamma = \begin{pmatrix} 1 \\ 1 \\ 5 \end{pmatrix}.$$

We check that

$$\begin{aligned} [U]_\beta^\gamma [h(x)]_\beta &= \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ 1 \\ 5 \end{pmatrix} = [U(h(x))]_\gamma. \end{aligned}$$

Problem 2.3.9. Find linear transformations U, $T : \mathbb{F}^2 \to \mathbb{F}^2$ such that $UT = T_0$ (the zero transformation) but $TU \neq T_0$. Use your answer to find matrices A and B such that $AB = 0$ but $BA \neq 0$.

Solution. Let $T(a,b) = (0, a+b)$ and let $U(a,b) = (a,a)$. Then

$$(UT)(a,b) = U(T(a,b)) = U(0, a+b) = (0,0)$$

and

$$(TU)(a,b) = T(U(a,b)) = T(a,a)) = (0, 2a).$$

So, we see that $UT = T_0$ and $TU \neq T_0$.

If β is the standard basis for $\mathbb{F}^2$, we compute

$$[T]_\beta = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \text{ and } [U]_\beta = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}.$$

Problem 2.3.11. Let V be a vector space, and let $T : V \to W$ be linear. Prove that $T^2 = T_0$ if and only if $R(T) \subseteq N(T)$.

Solution. *Proof.* Let V be a vector space, and let $T : V \to W$ be linear.

($\Rightarrow$) Suppose $T^2 = T_0$. Let $v \in V$.

$$T(T(v)) = T^2(v) = 0.$$

So, $T(v) \in N(T)$. This shows that $R(T) \subseteq N(T)$.

($\Leftarrow$) Suppose $R(T) \subseteq N(T)$. Let $v \in V$. Then

$$T^2(v) = T(T(v)) = 0,$$

since $T(v) \in R(T) \subseteq N(T)$. Since $v \in V$ is arbitrary, $T^2 = T_0$. □

Problem 2.3.13. Let A and B be $n \times n$ matrices. Recall that the trace of A is defined by

$$\operatorname{tr}(A) = \sum_{i=1}^{n} A_{ii}.$$

Prove that $\operatorname{tr}(AB) = \operatorname{tr}(BA)$ and $\operatorname{tr}(A) = \operatorname{tr}(A^t)$.

Solution. Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be $n \times n$ matrices. Then $AB = [c_{ij}]$ where

$$c_{ij} = \sum_{\ell=1}^{n} a_{i\ell} b_{\ell j}.$$

and $BA = [d_{ij}]$ where

$$d_{ij} = \sum_{\ell=1}^{n} b_{i\ell} a_{\ell j}.$$

Then

$$\begin{aligned}\operatorname{tr}(AB) = \sum_{i=1}^{n} c_{ii} &= \sum_{i=1}^{n} \sum_{k=1}^{n} a_{ik} b_{ki} \\ &= \sum_{k=1}^{n} \sum_{i=1}^{n} b_{ki} a_{ik} = \sum d_{ii} = \operatorname{tr}(BA).\end{aligned}$$

Finally, we note that $A^t = [a_{ji}]$, so $\operatorname{tr}(A^t) = \sum_{i=1}^{n} a_{ii} = \operatorname{tr}(A)$.