Problem Set #3 Solutions
Due Thursday, September 4

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Problem 1.6.14. Find bases for the following subspaces of $\mathbb{F}^5$:

$$W_1 = \{(a_1, a_2, a_3, a_4, a_5) \in \mathbb{F}^5 : a_1 - a_3 - a_4 = 0\}$$

and

$$W_2 = \{(a_1, a_2, a_3, a_4, a_5) \in \mathbb{F}^5 : a_2 = a_3 = a_4 \text{ and } a_1 + a_5 = 0\}.$$

What are the dimensions of W_1 and W_2?

Solution. One basis for W_1 is

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

The dimension of W_1 is four.

One basis for W_2 is

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} \right\}.$$

The dimension of W_2 is two.

Problem 1.6.15. The set of all $n \times n$ matrices having trace equal to zero is a subspace W of $M_{n\times n}(\mathbb{F})$. Find a basis for W. What is the dimension of W?

Solution. The dimension of W is $n^2 - 1$.

As usual, for $1 \leq i, j \leq n$, let E_{ij} be the matrix with 1 in the ij entry and zeroes elsewhere.

One basis for the vector space of $n \times n$ matrices having trace equal to zero is as follows:

$$\{E_{ij} \mid i \neq j\} \cup \{E_{ii} - E_{i+1,i+1} \mid 1 \leq i < n\}$$

For $n = 3$, this is the set

$$\left\{ \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \right\}$$

Problem 1.6.16. The set of all upper triangular $n \times n$ matrices is a subspace W of $M_{n\times n}(\mathbb{F})$. Find a basis for W. What is the dimension of W?

Solution. The dimension of W is $n(n+1)/2$.

As usual, for $1 \leq i, j \leq n$, let E_{ij} be the matrix with 1 in the ij entry and zeroes elsewhere.

One basis for the vector space of $n \times n$ matrices having trace equal to zero is as follows:

$$\{E_{ij} \mid 1 \leq i \leq j \leq n\}$$

For $n = 3$, this is the set

$$\left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \right\}$$

Problem 1.6.17. The set of all skew-symmetric $n \times n$ matrices is a subspace W of $M_{n\times n}(\mathbb{F})$. Find a basis for W. What is the dimension of W?

Solution. The dimension of W is $n(n-1)/2$.

As usual, for $1 \leq i, j \leq n$, let E_{ij} be the matrix with 1 in the ij entry and zeroes elsewhere. For $1 \leq i < j \leq n$, let $B_{ij} = E_{ij} - E_{ji}$

One basis for the vector space of $n \times n$ matrices having trace equal to zero is as follows:

$$\{B_{ij} \mid 1 \leq i < j \leq n\}$$

For $n = 3$, this is the set

$$\left\{ \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \right\}$$

Problem 2.1.5. Prove that

$$T : P_2(\mathbb{R}) \to P_3(\mathbb{R})$$
$$T(f(x)) = xf(x) + f'(x).$$

is a linear transformation, and find bases for both $N(T)$ and $R(T)$. Then compute the nullity and rank of T, and verify the dimension theorem. Finally, use the appropriate theorems in this section to determine whether T is one-to-one or onto.

Solution. Let $f(x)$ and $g(x) \in \mathbb{P}_2(\mathbb{R})$ and let $\lambda \in \mathbb{R}$. Then

$$\begin{aligned} T(f+g)(x)) &= x(f+g)(x) + (f+g)'(x) \\ &= x[f(x) + g(x)] + [f'(x) + g'(x)] \\ &= xf(x) + xg(x) + f'(x) + g'(x) \\ &= (xf(x) + f'(x)) + (xg(x) + g'(x)) \\ &= T(f) + T(g) \end{aligned}$$

and

$$\begin{aligned} T((\lambda f)(x)) &= T(\lambda f(x)) \\ &= x[\lambda f(x)] + [\lambda f(x)]' \\ &= \lambda[xf(x)] + \lambda f'(x) \\ &= \lambda[xf(x) + f'(x)] \\ &= \lambda T(f(x)). \end{aligned}$$

Since f, $g \in P_2(\mathbb{R})$ and $\lambda \in \mathbb{R}$ are arbitrary, T is a linear transformation.

The polynomial $ax^3 + bx^2 + cx + d$ lies in the image of T if and only if $b = d$. So, $\dim(R(T)) = 3$.

If $T(ax^2 + bx + c) = 0$, then $a = b = c = 0$, so T is injective and $\dim(N(T)) = 0$.

So, $\dim(N(T)) = 0$ and $\dim(R(T)) = 3$, so the dimension theorem is fulfilled.

Problem 2.1.6. Prove that

$$T : M_{n\times n}(\mathbb{F}) \to \mathbb{F}$$
$$T(A) = \operatorname{tr}(A) = \sum_{i=1}^{n} A_{ii}.$$

is a linear transformation, and find bases for both $N(T)$ and $R(T)$. Then compute the nullity and rank of T, and verify the dimension theorem. Finally, use the appropriate theorems in this section to determine whether T is one-to-one or onto.

Solution. Let M, $N \in M_{n\times n}(\mathbb{F})$ and a, $b \in \mathbb{F}$. By Exercise 6 in Section 1.3,

$$T(aM + bN) = aT(M) + bT(N).$$

Since M, $N \in M_{n\times n}(\mathbb{F})$ and a, $b \in \mathbb{F}$ are arbitrary, T is a linear transformation.

The null space of T is given by the single equation $\sum_{i=1}^{n} A_{ii} = 0$, so $\dim(N(T)) = n^2 - 1$. The function T is clearly surjective, so $\dim(R(T)) = 1$. So, $\dim(N(T)) = n^2 - 1$ and $\dim(R(T)) = 1$, so the dimension theorem is fulfilled.

Problem 2.1.14. Let V and W be vector spaces and $T : V \to W$ be linear.

(a) Prove that T is one-to-one if and only if T carries linearly independent subsets of V onto linearly independent subsets of W.

(b) Suppose that T is one-to-one and that S is a subset of V. Prove that S is linearly independent if and only if $T(S)$ is linearly independent.

(c) Suppose $\beta = \{v_1, v_2, \ldots, v_n\}$ is a basis for V and T is one-to-one and onto. Prove that $T(\beta) = \{T(v_1), T(v_2), \ldots, T(v_n)\}$ is a basis for W.

Solution. Let V and W be vector spaces and $T : V \to W$ be linear.

(a) *Proof.* ($\Rightarrow$) Suppose T is one-to-one. Let $T(v_1), \ldots, T(v_n) \in W$ for some linearly independent vectors $v_1, \ldots, v_n \in V$. Suppose $a_1T(v_1)+a_2T(v_2)+\cdots+a_nT(v_n) = 0$. Since T is linear, we have

$$T(a_1v_1 + a_2v_2 + \cdots a_nv_n) = 0,$$

and since T is one-to-one, this implies that

$$a_1v_1 + a_2v_2 + \cdots a_nv_n = 0.$$

Since $v_1, \ldots, v_n \in V$ are linear independent in V, we have $a_1 = a_2 = \cdots = a_n = 0$. It follows that $T(v_1), \ldots, T(v_n) \in W$ are linearly independent.

($\Leftarrow$) Suppose T carries linearly independent subsets of V onto linearly independent subsets. Suppose $v, w \in V$ with $v \neq w$. Then the vector $v - w \in V$ is nonzero and hence the set $\{v - w\}$ is linearly independent. By hypothesis, $\{T(v - w)\}$ is linearly independent, so $T(v-w) \neq 0$. That is $T(v) \neq T(w)$. This proves T is one-to-one. □

(b) *Proof.* Suppose T is one-to-one and S is a subset of V.

($\Rightarrow$) Suppose S is linearly independent. Let $T(v_1), T(v_2), \ldots, T(v_n) \in T(S)$ for some $v_1, \ldots, v_n \in S$ By part (a), $T(v_1), T(v_2), \ldots, T(v_n)$ are linearly independent. So, $T(S)$ is linearly independent.

($\Leftarrow$) Suppose $T(S)$ is linearly independent. By part (a), since $T(v_1), T(v_2), \ldots, T(v_n)$ are linearly independent, $v_1, v_2, \ldots, v_n$ are linearly independent. Hence, S is linearly independent.

□

(c) *Proof.* Suppose $\beta = \{v_1, v_2, \dots, v_n\}$ is a basis for V and T is one-to-one and onto. By part (b), $T(\beta) = \{T(v_1), T(v_2), \dots, T(v_n)\}$ is linearly independent.

Let $w \in W$ be arbitrary. Since T is onto, there exists $v \in V$ so that $T(v) = w$. Since β is a basis for V, we can write v uniquely as a linear combination

$$v = a_1 v_1 + a_2 v_2 + \cdots + a_n v_n.$$

But then

$$\begin{aligned} w = T(v) &= T(a_1 v_1 + a_2 v_2 + \cdots + a_n v_n) \\ &= a_1 T(v_1) + a_2 T(v_2) + \cdots + a_n T(v_n). \end{aligned}$$

Since $w \in W$ is arbitrary, $T(\beta)$ spans W.

Thus, $T(\beta)$ is a basis for W. □