Problem Set #2 Solutions
Due Thursday, August 28

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Problem 1.3.12. An $m \times n$ matrix A is called **upper triangular** if all entries lying below the diagonal entries are zero, that is, if $A_{ij} = 0$ whenever $i > j$. Prove that the upper triangular matrices form a subspace of $M_{m\times n}(\mathbb{F})$.

Solution. We only have to show that the set of upper triangular matrices is closed under the two operations of addition and scalar multiplication and that the zero matrix is upper triangular (which is clear).

For A, B upper triangular matrices and $\lambda \in \mathbb{F}$ we have

$$(A + B)_{ij} = A_{ij} + B_{ij} = 0 \text{ for } i > j$$

and

$$(\lambda A)_{ij} = \lambda A_{ij} = 0 \text{ for } i > j.$$

Since the set of upper triangular matrices is closed under the vector space operations (and contains the zero matrix), the set of upper triangular matrices form a subspace of $M_{m\times n}(\mathbb{F})$ by Theorem 1.3.

Problem 1.3.13. Let S a nonempty set and $\mathbb{F}$ a field. Prove that for any $s_0 \in S$, $\{f \in \mathcal{F}(S, \mathbb{F}) : f(s_0) = 0\}$, is a subspace of $\mathcal{F}(S, \mathbb{F})$.

Solution. Let S a nonempty set and $\mathbb{F}$ a field. Let

$$V = \{f \in \mathcal{F}(S, \mathbb{F}) : f(s_0) = 0\}.$$

Let f, $g \in V$ and let $\lambda \in \mathbb{F}$. Then

$$\begin{aligned}(f + g)(s_0) &= f(s_0) + g(s_0) = 0 + 0 = 0 \text{ and} \\ (\lambda f)(s_0) &= \lambda f((s_0) = \lambda \cdot 0 = 0.\end{aligned}$$

Also the zero function is in S. So V is a vector subspace of $\mathcal{F}(S, \mathbb{F})$ by Theorem 1.3.

Problem 1.3.20. Prove that if W is a subspace of a vector space V and $w_1, w_2, \ldots, w_n$ are in W, then $a_1w_1 + a_2w_2 + \cdots + a_nw_n \in W$ for any scalars $a_1, a_2, \ldots, a_n$.

Solution. *Proof.* Let V be a vector space and $W \subseteq V$ a subspace.

Let $w_1, w_2, \ldots, w_n \in W$ and $a_1, a_2, \ldots, a_n$ be scalars. Since W is a vector space, it is closed under scalar multiplication, so $a_iw_i \in W$ for all i, $1 \leq i \leq n$. Since W is a vector space, it is also closed under vector addition, so $\sum_{i=1}^{n} a_iw_i \in W$.

So, we see that $a_1w_1 + a_2w_2 + \cdots + a_nw_n \in W$. □

Problem 1.3.31. Let W be a subspace of a vector space V over a field $\mathbb{F}$. For any $v \in V$ the set $\{v\}+W = \{v+w : w \in W\}$ is called the **coset** of W **containing** v. It is customary to denote this coset by $v + W$ rather than $\{v\} + W$.

(a) Prove that $v + W$ is a subspace of V if and only if $v \in W$.

(b) Prove that $v_1 + W = v_2 + W$ if and only if $v_1 - v_2 \in W$.

Addition and scalar multiplication by scalars of $\mathbb{F}$ can be defined in the collection $S = \{v + W : v \in V\}$ of all cosets of W as follows:

$$(v_1 + W) + (v_2 + W) = (v_1 + v_2) + W$$

for all v_1, $v_2 \in V$ and

$$a(v + W) = av + W$$

for all $v \in V$ and $a \in \mathbb{F}$.

(c) Prove that the preceding operations are well defined; that is, show that if $v_1 + W = v_1' + W$ and $v_2 + W = v_2' + W$, then

$$(v_1 + W) + (v_2 + W) = (v_1' + W) + (v_2' + W)$$

and

$$a(v_1 + W) = a(v_1' + W)$$

for all $a \in \mathbb{F}$.

(d) Prove that the set S is a vector space with the operations defined in (c). This vector space is called the **quotient space of V modulo W** and is denoted by V/W.

Solution. Let W be a subspace of a vector space V over a field $\mathbb{F}$. For any $v \in V$ the set $\{v\} + W = \{v + w : w \in W\}$ is called the **coset** of W **containing** v. It is customary to denote this coset by $v + W$ rather than $\{v\} + W$.

(a) For $v \in V$, $v + W$ is a subspace if and only if it contains 0. $v + W$ contains 0 if and only there exists $w \in W$ so that $v + w = 0$. This happens if and only if $v = -w \in W$. So, $v + W$ is a subspace if and only if $v \in W$.

(b) Suppose $v_1 + W = v_2 + W$ for elements v_1, $v_2 \in V$. Then for some $w \in W$, $v_1 + 0 = v_2 + w$. That is $v_1 - v_2 \in W$. Conversely, suppose $v_1 - v_2 = w_0 \in W$. Then $v_1 + w = v_2 + (w_0 + w)$, so we see that $v_1 + W = v_2 + W$.

(c) Let $u + W = u' + W$ and $v + W = v' + W$. Then $u - u' \in W$ and $v - v' \in W$. Hence $(u + v) - (u' + v') = (u - u') + (v - v') \in W$. Thus, $(u + v) + W = (u' + v') + W$.

Let $v + W = v' + W$. Then $v - v' \in W$ so that $a(v - v') = av - av' \in W$. Thus $av + W = av' + W$.

(d) This is a routine verification of the eight defining properties of a vector space. The proof uses only the fact that V itself is a vector space and the defining properties there.

Problem 1.4.10. Show that if

$$M_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \text{ and } M_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

then the span of $\{M_1, M_2, M_3\}$ is the set of all symmetric 2×2 matrices.

Solution. By definition, the span of the set $\{M_1, M_2, M_3\}$ is the set of all matrices of the form

$$aM_1 + bM_2 + cM_3 = a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + c\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} a & b \\ b & c \end{pmatrix}.$$

where a, b, and $c \in \mathbb{F}$.

We see this is exactly the set of symmetric 2×2 matrices.

Problem 1.5.6. In $M_{m\times n}(\mathbb{F})$, let E^{ij} denote the matrix whose only nonzero entry is 1 in the ith row and jth column. Prove that $\{E^{ij} : 1 \leq i \leq m, 1 \leq j \leq n\}$ is linearly independent.

Solution. *Proof.* Suppose $\sum \lambda_{ij} E^{ij} = \mathbf{0}$, the zero matrix. We compute that this sum is

$$\begin{pmatrix} \lambda_{11} & \lambda_{12} & \cdots & \lambda_{1n} \\ \lambda_{21} & \lambda_{22} & \cdots & \lambda_{2n} \\ \vdots & \vdots & & \vdots \\ \lambda_{m1} & \lambda_{m2} & \cdots & \lambda_{mn} \end{pmatrix}.$$

This is the zero matrix if and only if $\lambda_{ij} = 0$ for all i, j.

It follows that the set $\{E^{ij} : 1 \leq i \leq m, 1 \leq j \leq n\}$ is linearly independent. □

Problem 1.5.9. Let u and v be distinct vectors in a vector space V. Show that $\{u, v\}$ is linearly dependent if and only if u or v is a multiple of the other.

Solution. *Proof.* Let u and v be distinct vectors in a vector space V.

($\Rightarrow$) Suppose $\{u, v\}$ is linearly dependent. Then there exist scalars λ_1, $\lambda_2 \in \mathbb{F}$, not both zero, so that

$$\lambda_1 u + \lambda_2 v = 0.$$

If $\lambda_1 \neq 0$, we have $u = -(\lambda_2/\lambda_1)v$, whereas if $\lambda_2 \neq 0$, we have $v = -(\lambda_1/\lambda_2)u$. So, u or v is a multiple of the other.

($\Leftarrow$) Conversely, suppose u or v is a multiple of the other. If $u = \lambda v$ for some $\lambda \in \mathbb{F}$, then $1u - \lambda v = 0$, so $\{u, v\}$ is linearly dependent. If $v = \lambda u$ for some $\lambda \in \mathbb{F}$, then $\lambda u - 1v = 0$, so again $\{u, v\}$ is linearly dependent. $\square$

Problem 1.5.10. Given an example of three linearly dependent vectors in $\mathbb{R}^3$ such that none of the three is a multiple of another.

Solution.

Example. Let $v_1 = \mathbf{e}_1$, $v_2 = \mathbf{e}_2$, and $v_3 = \mathbf{e}_1 + \mathbf{e}_2$. Then

$$v_1 + v_2 - v_3 = 0$$

but none of $v_1 = \mathbf{e}_1$, $v_2 = \mathbf{e}_2$, or $v_3 = \mathbf{e}_1 + \mathbf{e}_2$ is a multiple of another.

Problem 1.5.14. Prove that a set S is linearly dependent if and only if $S = \{0\}$ or there exist distinct vectors v, u_1, u_2, $\ldots$, u_n in S such that v is a linear combination of u_1, u_2, $\ldots$, u_n.

Solution. *Proof.* Let S be a set of vectors in a vector space V.

($\Rightarrow$) Suppose S is linearly dependent. We assume $S \neq \{0\}$.

Suppose S contains 0. By the definition of linear dependence, there exist vectors u_1, $\ldots$, $u_n \in S$, and scalars λ_1, $\ldots$, λ_n, not all zero, so that

$$\lambda_1 u_1 + \lambda_2 u_2 + \cdots + \lambda_n u_n = 0. \tag{1}$$

By combining terms if necessary that all the u_i's are distinct. Setting $v = 0$, we get

$$v = \lambda_1 u_1 + \lambda_2 u_2 + \cdots + \lambda_n u_n.$$

So, we see there exist distinct vectors v, u_1, u_2, $\ldots$, u_n in S such that v is a linear combination of u_1, u_2, $\ldots$, u_n.

Now suppose $0 \notin S$. Since S is linearly dependent, there exist vectors u_1, $\ldots$, u_n, u_{n+1} and scalars λ_1, $\ldots$, λ_n, λ_{n+1}, not all zero, so that

$$\lambda_1 u_1 + \lambda_2 u_2 + \cdots + \lambda_n u_n + \lambda_{n+1} u_{n+1} = 0. \tag{2}$$

By renumbering the terms if necessary, we may assume that $\lambda_{n+1} \neq 0$. By combining terms, we may also assume all the u_i's are distinct. So, we have Equation (2) with distinct vectors and $\lambda_{n+1} \neq 0$. Letting $v = u_{n+1}$ and solving for v, we have

$$v = -\left[\left(\frac{\lambda_1}{\lambda_{n+1}}\right) u_1 + \left(\frac{\lambda_2}{\lambda_{n+1}}\right) u_2 + \cdots + \left(\frac{\lambda_n}{\lambda_{n+1}}\right) u_n\right].$$

So, we see there exist distinct vectors v, u_1, u_2, $\ldots$, u_n in S such that v is a linear combination of u_1, u_2, $\ldots$, u_n.

We conclude that $S = \{0\}$ or there exist distinct vectors v, u_1, u_2, $\ldots$, u_n in S such that v is a linear combination of u_1, u_2, $\ldots$, u_n.

($\Leftarrow$) First, if $S = \{0\}$ it is linearly dependent and we're done. So, we assume $S \neq \{0\}$. Suppose there exist distinct vectors v, u_1, u_2, $\ldots$, u_n in S such that v is a linear combination of u_1, u_2, $\ldots$, u_n. Then there exists scalars λ_1, λ_2, $\ldots$, λ_n so that

$$v = \lambda_1 u_1 + \lambda_2 u_2 + \cdots + \lambda_n u_n.$$

Then

$$0 = \lambda_1 u_1 + \lambda_2 u_2 + \cdots + \lambda_n u_n - v,$$

so S is linearly dependent. □