Problem Set #1 Solutions
Due Thursday, August 21

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Problem 1.1.1. Determine whether the vectors emanating from the origin and terminating at the following pairs of points are parallel.

(a) $(3, 1, 2)$ and $(6, 4, 2)$

(b) $(-3, 1, 7)$ and $(9, -3, -21)$

(c) $(5, -6, 7)$ and $(-5, 6, -7)$

(d) $(2, 0, -5)$ and $(5, 0, -2)$

Solution. (a) The vectors $\langle 3, 1, 2\rangle$ and $\langle 6, 4, 2\rangle$ are not parallel.

(b) The vectors $\langle -3, 1, 7\rangle$ and $\langle 9, -3, -21\rangle$ are multiples of each other, so they are parallel.

(c) The vectors $\langle 5, -6, 7\rangle$ and $\langle -5, 6, -7\rangle$ are multiples of each other, so they are parallel.

(d) The vectors $\langle 2, 0, -5\rangle$ and $\langle 5, 0, -2\rangle$ are not parallel.

Problem 1.1.2. Find the equations of the lines through the following pairs of points in space.

(a) $(3, -2, 4)$ and $(-5, 7, 1)$

Solution. The vector from $(3, -2, 4)$ to $(-5, 7, 1)$ is $\langle -5, 7, 1\rangle - \langle 3, -2, 4\rangle = \langle -8, 9, -3\rangle$.

The equation of the line through the points $(3, -2, 4)$ and $(-5, 7, 1)$ is

$$\mathbf{x} = \langle 3, -2, 4\rangle + t\langle -8, 9, -3\rangle.$$

Problem 1.1.3. Find the equations of the plane containing the following points in space.

(a) $(2,-5,-1)$, $(0,4,6)$, and $(-3,7,1)$

Solution. The vector from $(2,-5,-1)$ to $(0,4,6)$ is $\langle 0,4,6\rangle - \langle 2,-5,-1\rangle = \langle -2,9,7\rangle$. The vector from $(2,-5,-1)$ to $(-3,7,1)$ is $\langle -3,7,1\rangle - \langle 2,-5,-1\rangle = \langle -5,12,2\rangle$.

The equation of the plane through the points $(2,-5,-1)$, $(0,4,6)$, and $(-3,7,1)$ is

$$\mathbf{x} = \langle 2,-5,-1\rangle + s\langle -2,9,7\rangle + t\langle -5,12,2\rangle.$$

Problem 1.1.6. Show that the midpoint of the line segment joining the points (a, b) and (c, d) is $((a + c)/2, (b + d)/2)$.

Solution. The position vector for the point (a, b) is $\langle a, b\rangle$. The position vector for the point (c, d) is $\langle c, d\rangle$. The vector from (a, b) to (c, d) is $\langle c - a, d - b\rangle$.

The position vector for the midpoint of the line segment joining the points (a, b) and (c, d) is then

$$\langle a, b\rangle + \frac{1}{2}\langle c - a, d - b\rangle = \left\langle a + \frac{1}{2}(c - a), b + \frac{1}{2}(d - b)\right\rangle = \left\langle \frac{1}{2}(a + c), \frac{1}{2}(b + d)\right\rangle.$$

So, the midpoint of the segment is the point $\left(\frac{1}{2}(a + c), \frac{1}{2}(b + d)\right)$.

Problem 1.2.10. Let V denote the set of all differentiable real-valued functions defined on the real line. Prove that V is a vector space with the operations of addition and scalar multiplication defined in Example 3.

Proof. Let V denote the set of all differentiable real-valued functions defined on the real line.

We first show that V is closed under addition and scalar multiplication. If f and g are differentiable real-valued functions defined on the real line, we know (from Calculus 1) $f + g$ is differentiable real-valued functions defined on the real line, so the sum lies in V. For a scalar a, we know (from Calculus 1) af is differentiable real-valued functions defined on the real line. So, $af \in V$.

We now must show that V with the operations of addition of functions and scalar multiplication of functions satisfies Properties (VS 1)–(VS 8).

Let $x, y, z \in V$. Since the addition of real numbers is both commutative and associative, we have $x + y = y + x$ and $(x + y) + z = x + (y + z)$.

The zero function 0 which assigns 0 to each element of $\mathbb{R}$ is differentiable and $x + 0 = 0 + x = x$.

Since the function x is differentiable, so is the function $-x$, and $x+(-x) = (-x)+x = 0$.

The function 1 which assigns 1 to each element of $\mathbb{R}$ is differentiable and $1x = x$.

The remaining properties follow since the real numbers have associative property of multiplication and the distributive property of multiplication over addition. □

Problem 1.2.12. A real-valued function f defined on the real line is called an **even function** if $f(-t) = f(t)$ for each real number t. Prove that the set of even functions defined on the real line with the operations of addition and scalar multiplication defined in Example 3 is a vector space.

Proof. Let V be the set of all even functions on the real line.

We first verify that V is closed under addition and scalar multiplication.

Let f, $g \in V$ and $a \in \mathbb{R}$. Then for $s \in \mathbb{R}$, we have

$$(f+g)(-s) = f(-s) + g(-s) = f(s) + g(s) = (f+g)(s)$$

and

$$(af)(-s) = af(-s) = af(s) = (af)(s).$$

So, we see that $f + g$ and af are even functions and are therefore in V.

To show V is a real vector space we need only show that V with the operations of addition of functions and scalar multiplication of functions satisfies Properties (VS 1)–(VS 8). This is done exactly as in the last proof. □

Problem 1.2.14. Let $V = \{(a_1, a_2, \ldots, a_n) : a_i \in \mathbb{C} \text{ for } 1 \leq i \leq n\}$. So V is a vector space over $\mathbb{C}$ by Example 1. Is V a vector space over the field of real numbers with the operations of coordinatewise addition and multiplication?

Solution. Since the product of a real number and a complex number is a complex number, V is a vector space over $\mathbb{R}$.

Problem 1.2.22. How many matrices are there in the vector space $M_{m\times n}(\mathbb{Z}_2)$? (See Appendix C.)

Solution. Since $\mathbb{Z}_2$ contains two elements, there are two choices for each entry in an $m \times n$ matrix, so the number of elements in $M_{m\times n}(\mathbb{Z}_2)$ is 2^{mn}.