Problem Set #14 Solutions
Due Thursday, November 20

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Problem 6.3.2. For each of the following inner product spaces V (over $\mathbb{F}$) and linear transformations $g : V \to \mathbb{F}$, find a vector y such that $g(x) = \langle x, y\rangle$ for all $x \in V$.

(a) $V = \mathbb{R}^3$, $g(a_1, a_2, a_3) = a_1 - 2a_2 + 4a_3$

Solution. (a)

$$g(a_1, a_2, a_3) = a_1 - 2a_2 + 4a_3 = \langle (a_1, a_2, a_3), (1, -2, 4)\rangle.$$

So, $y = (1, -2, 4)$.

Problem 6.3.6. Let T be a linear operator on an inner product space V. Let $U_1 = T + T^*$ and $U_2 = TT^*$. Prove that $U_1 = U_1^*$ and $U_2 = U_2^*$.

Solution. *Proof.* Let T be a linear operator on an inner product space V. Let $U_1 = T + T^*$ and $U_2 = TT^*$.

Then

$$\begin{aligned} U_1^* &= (T + T^*)^* \\ &= T^* + T^{**} \\ &= T^* + T \\ &= T + T^* \\ &= U_1. \end{aligned}$$

and

$$\begin{aligned} U_2^* &= (TT^*)^* \\ &= T^{**}T^* \\ &= TT^* \\ &= U_2. \end{aligned}$$

□

Problem 6.3.9. Prove that if $V = W \oplus W^{\perp}$ and T is projection on W along $W^{\perp}$, then $T = T^*$. *Hint:* Recall that $N(T) = W^{\perp}$. (For definitions, see the exercises of Sections 1.3 and 2.1.)

Solution. Let V be an inner product space and $W \subseteq V$ a subspace. Let $T : V \to V$ be projection of V along $W^{\perp}$. Let x, $y \in V$ and write $x = w_1 + z_1$, $y = w_2 + z_2$ with w_1, $w_2 \in W$ and z_1, $z_2 \in W^{\perp}$.

$$\begin{aligned}\langle T(x), y\rangle &= \langle w_1, w_2 + z_2\rangle = \langle w_1, w_2\rangle + \langle w_1, z_2\rangle = \langle w_1, w_2\rangle \\ &= \langle w_1, w_2\rangle + \langle z_1, w_2\rangle = \langle w_1 + z_1, w_2\rangle = \langle x, T(y)\rangle.\end{aligned}$$

So, we see $T = T^*$.

Problem 6.3.11. For a linear operator T on an inner product space V, prove that $T^*T = T_0$ implies $T = T_0$. Is the same result true if we assume that $TT^* = T_0$?

Solution. Let T be a linear operator on an inner product space V satisfying $T^*T = T_0$. Let $x \in V$ be arbitrary. Then

$$0 = \langle 0, x\rangle = \langle T_0(x), x\rangle = \langle T^*T(x), x\rangle = \langle T(x), T(x)\rangle = \|T(x)\|^2.$$

which implies $T(x) = 0$. Since $x \in V$ is arbitrary, $T = T_0$.

Suppose $TT^* = T_0$. By the last paragraph, $T^* = T_0$. Then

$$0 = \langle x, 0\rangle = \langle x, T_0(T(x))\rangle = \langle x, T^*(T(x))\rangle = \langle T(x), T(x)\rangle,$$

and since $x \in V$ is arbitrary, this says $T = T_0$.

Problem 6.3.12. Let V be an inner product space, and let T be a linear operator on V. Prove the following results.

(a) $R(T^*)^\perp = N(T)$.

(b) If V is finite-dimensional, then $R(T^*) = N(T)^\perp$. *Hint:* Use Exercise 13(c) of Section 6.2.

Solution. Let V be an inner product space, and let T be a linear operator on V.

(a) *Proof.* Let $x \in R(T^*)^\perp$. Since $x \in R(T^*)^\perp$,

$$0 = \langle x, T^*(T(x)) \rangle = \langle T(x), T(x) \rangle$$

so $T(x) = 0$ and $x \in N(T)$. Since $x \in R(T^*)^\perp$ is arbitrary, $R(T^*)^\perp \subseteq N(T)$.

On the other hand, let $x \in N(T)$ and let $y = T^*(z) \in R(T^*)$ be arbitrary. Then

$$0 = \langle 0, z \rangle = \langle T(x), z \rangle = \langle x, T^*(z) \rangle = \langle x, y \rangle.$$

Since $y = T^*(z) \in R(T^*)$ is arbitrary, $x \in R(T^*)^\perp$. Since $x \in N(T)$ is arbitrary, $N(T) \subseteq R(T^*)^\perp$.

The two inclusions prove that $R(T^*)^\perp = N(T)$.

□

(b) *Proof.* Suppose in addition that V is finite dimensional. Then $R(T)$ is finite dimensional, so by Exercise 13(c) of Section 6.2, $(R(T^*)^\perp)^\perp = R(T^*)$. Taking the orthogonal complement of the result in part(a), we see that

$$R(T^*) = (R(T^*)^\perp)^\perp = N(T)^\perp.$$

□

Problem 6.3.18. Let A be an $n \times n$ matrix. Prove that $\det(A^*) = \overline{\det(A)}$.

Solution. Let A be an $n \times n$ matrix. We compute

$$\begin{aligned} \det(A^*) &= \det\left(\overline{A^t}\right) \\ &= \overline{\det(A^t)} \\ &= \overline{\det(A)}. \end{aligned}$$

Problem 6.4.3. Give an example of a linear operator T on $\mathbb{R}^2$ and an ordered basis for $\mathbb{R}^2$ that provides a counterexample to the statement in Exercise 1(c).

Solution. We first give the statement in Exercise 1(c):

If T is an operator on an inner product space V, then T is normal if and only if $[T]_\beta$ is normal, where β is any ordered basis for V.

This is true for orthonormal bases, but not true in general.

Let $\beta = \{(1,1),(1,0)\}$ be a basis for $\mathbb{R}^2$ and let $T(a,b) = (2a,b)$. Then T is normal with $T^* = T$. But $[T]_\beta = \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}$ is not normal.

Problem 6.4.4. Let T and U be self-adjoint operators on an inner product space V. Prove that TU is self-adjoint if and only if $TU = UT$.

Solution. *Proof.* Let T and U be self-adjoint operators on an inner product space V.

($\Rightarrow$) Suppose TU is self-adjoint. Then $TU = (TU)^* = U^*T^* = UT$. So, $TU = UT$.

($\Leftarrow$) Suppose $TU = UT$. Then $(TU)^* = U^*T^* = UT = TU$, so TU is self-adjoint. $\square$

Problem 6.4.6. Let V be a complex inner product space, and let T be a linear operator on V. Define

$$T_1 = \frac{1}{2}\left(T + T^*\right) \text{ and } T_2 = \frac{1}{2i}\left(T - T^*\right).$$

(a) Prove that T_1 and T_2 are self-adjoint and that $T = T_1 + iT_2$.

(b) Suppose also that $T = U_1 + iU_2$, where U_1 and U_2 are self-adjoint. Prove that $U_1 = T_1$ and $U_2 = T_2$.

(c) Prove that T is normal if and only if $T_1T_2 = T_2T_1$.

Solution. Let V be a complex inner product space, and let T be a linear operator on V. Define

$$T_1 = \frac{1}{2}\left(T + T^*\right) \text{ and } T_2 = \frac{1}{2i}\left(T - T^*\right).$$

(a) We compute

$$\begin{aligned} T_1^* &= \left[\frac{1}{2}\left(T + T^*\right)\right]^* \\ &= \frac{1}{2}\left(T + T^*\right)^* \\ &= \frac{1}{2}\left(T^* + T^{**}\right) \\ &= \frac{1}{2}\left(T^* + T\right) \\ &= T_1. \end{aligned}$$

and

$$\begin{aligned} T_2^* &= \left[\frac{1}{2i}\left(T - T^*\right)\right]^* \\ &= -\frac{1}{2i}\left[\left(T - T^*\right)\right]^* \\ &= -\frac{1}{2i}\left(T^* - T^{**}\right) \\ &= -\frac{1}{2i}\left(T^* - T\right) \\ &= \frac{1}{2i}\left(T - T^*\right) \\ &= T_2. \end{aligned}$$

So, T_1 and T_2 are self-adjoint.

(b) Suppose also that $T = U_1 + iU_2$, where U_1 and U_2 are self-adjoint. We remark that $T^* = U_1 - iU_2$, since U_1 and U_2 are self-adjoint.

Then from part (a), we know that

$$\begin{aligned} T_1 &= \frac{1}{2}(T + T^*) \\ &= \frac{1}{2}[(U_1 + iU_2) + (U_1 - iU_2)] \\ &= U_1 \end{aligned}$$

and

$$\begin{aligned} T_2 &= \frac{1}{2i}(T - T^*) \\ &= \frac{1}{2i}[(U_1 + iU_2) - (U_1 + iU_2)] \\ &= U_2. \end{aligned}$$

(c) Suppose T is normal and let T_1 and T_2 be as above. We remark that since T_1 and T_2 are self-adjoint (as shown in (a) above), $T^* = T_1 - iT_2$.

Since T is normal, we have

$$\begin{aligned} TT^* &= T^*T \\ (T_1 + iT_2)(T_1 - iT_2) &= (T_1 - iT_2)(T_1 + iT_2) \\ T_1T_1 - iT_1T_2 + iT_2T_1 + T_2T_2 &= T_1T_1 + iT_1T_2 - iT_2T_1 + T_2T_2 \\ -2iT_1T_2 &= -2iT_2T_1 \\ T_1T_2 &= T_2T_1. \end{aligned}$$

For the converse, simply start at the bottom and work your way back to the top.

Problem 6.4.7. Let T be a linear operator on an inner product space V, and let W be a T-invariant subspace of V. Prove that following results.

(a) If T is self-adjoint, then T_W is self-adjoint.

(b) $W^\perp$ is T^*-invariant.

(c) If W is both T- and T^*-invariant, then $(T_W)^* = (T^*)_W$.

(d) If W is both T- and T^*-invariant and T is normal, then T_W is normal.

Solution. Let T be a linear operator on an inner product space V, and let W be a T-invariant subspace of V.

(a) Suppose T is self-adjoint and w_1, $w_2 \in W$. Then

$$\begin{aligned}\langle T_W(w_1), w_2\rangle &= \langle T(w_1), w_2\rangle \\ &= \langle w_1, T(w_2)\rangle \\ &= \langle w_1, T_W(w_2)\rangle\end{aligned}$$

So, $(T_W)^* = T_W$.

(b) Let $x \in W^\perp$ and let $w \in W$. Then

$$\langle T^*(x), w\rangle = \langle x, T(w)\rangle = 0,$$

since W is T-invariant by hypothesis. Since $w \in W$ is arbitrary, $T^*(x) \in W^\perp$. Since $x \in W^\perp$ is arbitrary, $W^\perp$ is T^*-invariant.

(c) Suppose W is both T- and T^*-invariant. Let w_1, $w_2 \in W$. Then

$$\begin{aligned}\langle T_W(w_1), w_2\rangle &= \langle T(w_1), w_2\rangle \\ &= \langle w_1, T^*(w_2)\rangle.\end{aligned}$$

Since $T^*(w_2) \in W$, $T^*(w_2) = T^*_W(w_2)$. So, we see that $(T_W)^* = T^*_W$

(d) Suppose W is both T- and T^*-invariant and T is normal. Then $T^*T = TT^*$. Restricting to W, remembering that W is both T- and T^* invariant and part (c) above, we get

$$(T_W)^*T_W = T^*_W T_W = (T^*T)_W = (TT^*)_W = T_W T^*_W = T_W(T_W)^*,$$

so T_W is normal.

Problem 6.4.8. Let T be a normal operator on a finite-dimensional complex inner product space V, and let W be a subspace of V. Prove that if W is T-invariant, then W is also T^*-invariant. *Hint:* Use Exercise 24 of section 5.4.

Solution. Let T be a normal operator on a finite-dimensional complex inner product space V, and let W be a T-invariant subspace of V.

By Theorem 6.16, there exists an orthonormal basis for V consisting of eigenvectors of T. So, T is diagonalizable. By Exercise 24 in Section 5.4, T_W is also diagonalizable. So there exists an orthonormal basis $\beta = \{w_1, \ldots, w_k\}$ for W of eigenvectors for T. By Theorem 6.15 (c), every eigenvector for T is also an eigenvector for T^*. Hence $\beta = \{w_1, \ldots, w_k\}$ is a basis for W of eigenvectors for T^*. So, W is T^*-invariant.