Problem Set #13 Solutions
Due Thursday, November 13

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Problem 6.1.4. (a) Complete the proof in Example 5 that $(\cdot, \cdot)$ is an inner product (the Frobenius inner product) on $M_{n\times n}(\mathbb{F})$.

Solution. (a) We prove the formula

$$\begin{aligned}\langle A, B\rangle = \operatorname{tr}(B^*A) &= \sum_{j=1}^{n}(B^*A)_{jj} = \sum_{j=1}^{n}\sum_{i=1}^{n} B^*_{ji}A_{ij} \\ &= \sum_{j=1}^{n}\sum_{i=1}^{n} A_{ij}\overline{B}_{ij} = \sum_{i=1}^{n}\sum_{j=1}^{n} A_{ij}\overline{B}_{ij} = \sum_{i,j} A_{ij}\overline{B}_{ij}.\end{aligned}$$

So we may view the space $M_{n\times n}(\mathbb{F})$ to be $\mathbb{F}^{n^2}$ and the Frobenius inner product is corresponding to the standard inner product in $\mathbb{F}^{n^2}$.

Problem 6.1.9. Let β be a basis for a finite-dimensional inner product space.

(a) Prove that if $\langle x, z\rangle = 0$ for all $z \in \beta$, then $x = 0$.

(b) Prove that if $\langle x, z\rangle = \langle y, z\rangle$ for all $z \in \beta$, then $x = y$.

Solution. Let β be a basis for a finite-dimensional inner product space.

(a) *Proof.* Suppose $\langle x, z\rangle = 0$ for all $z \in \beta$. Since β is a basis, we can write $x = \sum_{i=1}^{n} x_i \mathbf{v}_i$, where $\mathbf{v}_i \in \beta$ and $x_i \in \mathbb{F}$ for all $1 \le i \le n$. Then

$$\begin{aligned}\langle x, x\rangle &= \langle x, \sum_{i=1}^{n} x_i \mathbf{v}_i\rangle \\ &= \sum_{i=1}^{n} \overline{x_i}\langle x, \mathbf{v}_i\rangle \\ &= \sum_{i=1}^{n} \overline{x_i} \cdot 0 \\ &= 0,\end{aligned}$$

since $\langle x, \mathbf{v}_i\rangle = 0$ for all $1 \le i \le n$. Hence $x = 0$. □

(b) *Proof.* Suppose $\langle x, z\rangle = \langle y, z\rangle$ for all $z \in \beta$. Then

$$\langle x - y, z\rangle = \langle x, z\rangle - \langle y, z\rangle = 0$$

for all $z \in \beta$. By part (a), $x - y = 0$, so $x = y$. □

Problem 6.1.10. Let V be an inner product space, and suppose that x and y are orthogonal vectors in V. Prove that $\|x+y\|^2 = \|x\|^2 + \|y\|^2$. Deduce the Pythagorean theorem in $\mathbb{R}^2$.

Solution. *Proof.* Let V be an inner product space, and suppose that x and y are orthogonal vectors in V. Then

$$\begin{aligned}\|x+y\|^2 &= \langle x+y, x+y\rangle \\ &= \langle x,x\rangle + \langle x,y\rangle + \langle y,x\rangle + \langle y,y\rangle \\ &= \langle x,x\rangle + \langle y,y\rangle \text{ since } x \text{ and } y \text{ are orthogonal} \\ &= \|x\|^2 + \|y\|^2.\end{aligned}$$

If x and y are the two legs of a right triangle with lengths a and b, respectively, then $x+y$ is the hypotenuse of the right triangle. Let c be the length of the hypotenuse. Then the result above shows that

$$a^2 + b^2 = \|x\|^2 + \|y\|^2 = \|x+y\|^2 = c^2,$$

the Pythagorean theorem. □

Problem 6.1.11. Prove the *parallelogram law* on an inner product space V; that is, show that

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2 \text{ for all } x, y \in V$$

What does this equation state about parallelograms in $\mathbb{R}^2$?

Solution. Let V be an inner product space and let $x,\ y \in V$. Then

$$\begin{aligned}
\|x+y\|^2 + \|x-y\|^2 &= \langle x+y, x+y\rangle + \langle x-y, x-y\rangle \\
&= \langle x,x\rangle + \langle x,y\rangle + \langle y,x\rangle + \langle y,y\rangle \\
&\quad\ \langle x,x\rangle - \langle x,y\rangle - \langle y,x\rangle + \langle y,y\rangle \\
&= 2\|x\|^2 + 2\|y\|^2.
\end{aligned}$$

This equality says that the sum of the squares of the lengths of the four sides of a parallelogram equals the sum of the squares of the lengths of the two diagonals of the parallelogram.

Problem 6.1.12. Let $\{v_1, v_2, \ldots, v_k\}$ be an orthogonal set in V, and let $a_1, a_2, \ldots, a_k$ be scalars. Prove that

$$\left\|\sum_{i=1}^{k} a_i v_i\right\|^2 = \sum_{i=1}^{k} |a_i|^2 \, \|v_i\|^2.$$

Solution. We compute

$$\begin{aligned}
\left\|\sum_{i=1}^{k} a_i v_i\right\|^2 &= \left\langle \sum_{i=1}^{k} a_i v_i, \sum_{j=1}^{k} a_j v_j \right\rangle \\
&= \sum_{i=1}^{k} \sum_{j=1}^{k} a_i \overline{a_j} \langle v_i, v_j \rangle \\
&= \sum_{i=1}^{k} a_i \overline{a_i} \langle v_i, v_i \rangle \\
&= \sum_{i=1}^{k} |a_i|^2 \, \|v_i\|^2.
\end{aligned}$$

Problem 6.2.6. Let V be an inner product space, and let W be a finite-dimensional subspace of V. If $x \notin W$, prove that there exists $y \in V$ such that $y \in W^{\perp}$, but $\langle x, y \rangle \neq 0$. *Hint:* Use Theorem 6.6.

Solution. *Proof.* Let V be an inner product space, and let W be a finite-dimensional subspace of V. Let $x \in V$, but $x \notin W$. By Theorem 6.6, we can write x uniquely as $x = w + y$ where $w \in W$ and $y \in W^{\perp}$. Since $x \notin W$, $y \neq 0$. Then we have

$$\langle x, y \rangle = \langle w + y, y \rangle = \langle w, y \rangle + \langle y, y \rangle = \langle y, y \rangle > 0,$$

since $y \neq 0$. □

Problem 6.2.10. Let W be a finite-dimensional subspace of an inner product space V. Prove that there exists a projection T on W along $W^{\perp}$ that satisfies $N(T) = W^{\perp}$. In addition, prove that $\|T(x)\| \leq \|x\|$ for all $x \in V$. *Hint:* Use Theorem 6.6 and Exercise 10 of Section 6.1. (Projections are defined in the exercises of Section 2.1.)

Solution. *Proof.* Let W be a finite-dimensional subspace of an inner product space V. For $x \in V$, by Theorem 6.6, there exist unique vectors $u \in W$ and $z \in W^{\perp}$ so that $x = u + z$. Define $T : V \to V$ by $T(x) = u$. It's easy to see that $N(T) = W^{\perp}$. Also, by Exercise 10 of Section 6.1, we have

$$\|T(x)\|^2 = \|u\|^2 \leq \|u\|^2 + \|z\|^2 = \|u + z\|^2 = \|x\|^2.$$

Taking square roots and remembering that everything is positive, we get

$$\|T(x)\| \leq \|x\|.$$

$\square$

Problem 6.2.15. Let V be a finite-dimensional inner product space over $\mathbb{F}$.

(a) *Parseval's Identity.* Let $\{v_1, v_2, \ldots, v_n\}$ be an orthonormal basis for V. For any x, $y \in V$ prove that

$$\langle x, y\rangle = \sum_{i=1}^{n} \langle x, v_i\rangle \overline{\langle y, v_i\rangle}.$$

(b) Use (a) to prove that if β is an orthonormal basis for V with inner product $\langle \cdot, \cdot\rangle$, then for any x, $y \in V$

$$\langle \phi_\beta(x), \phi_\beta(y)\rangle' = \langle [x]_\beta, [y]_\beta\rangle' = \langle x, y\rangle,$$

where $\langle \cdot, \cdot\rangle'$ is the standard inner product on $\mathbb{F}^n$.

Solution. (a) Let $\{v_1, v_2, \ldots, v_n\}$ be an orthonormal basis for V. Let x, $y \in V$. By Theorem 6.6, we can write

$$x = \sum_{i=1}^{n} \langle x, v_i\rangle\, v_i$$
$$y = \sum_{i=1}^{n} \langle y, v_i\rangle\, v_i.$$

For convenience, we denote $x_i = \langle x, v_i\rangle$ and $y_i = \langle y, v_i\rangle$ for $1 \leq i \leq n$.

Then

$$\begin{aligned}
\langle x, y\rangle &= \left\langle \sum_{i=1}^{n} x_i v_i, \sum_{j=1}^{n} y_j v_j \right\rangle \\
&= \sum_{i=1}^{n} x_i \sum_{j=1}^{n} \overline{y_j}\, \langle v_i, v_j\rangle \\
&= \sum_{i=1}^{n} x_i \sum_{j=1}^{n} \overline{y_j}\, \delta_{i,j} \\
&= \sum_{i=1}^{n} x_i \overline{y_i} \\
&= \sum_{i=1}^{n} \langle x, v_i\rangle \overline{\langle y, v_i\rangle}.
\end{aligned}$$

(b) Let β be an orthonormal basis for V with inner product $\langle \cdot, \cdot\rangle$. For any $x \in V$, write $x = \sum_{i=1}^{n} x_i v_i$ with $v_i \in \beta$, $1 \leq i \leq n$. We note that as in part (a), we have $x_i = \langle x, v_i\rangle$ for $1 \leq i \leq n$. Define

$$\phi_\beta : V \to \mathbb{F}^n \quad \text{by} \quad \phi_\beta(x) = (x_1, x_2, \ldots, x_n).$$

If $\langle \cdot, \cdot \rangle'$ is the standard inner product on $\mathbb{F}^n$, we compute

$$\begin{aligned} \langle \phi_\beta(x), \phi_\beta(y) \rangle' &= \langle [x]_\beta, [y]_\beta \rangle' \\ &= \sum_{i=1}^{n} x_i \overline{y_i} \\ &= \langle x, y \rangle \text{ by part (a).} \end{aligned}$$