# Problem Set #12 Solutions
# Due Thursday, November 6

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**Problem 5.2.2.** For each of the following matrices $A \in M_{n\times n}(\mathbb{R})$, test $A$ for diagonalizability, and if $A$ is digonalizable, find on invertible matrix $Q$ and a diagonal matrix $D$ such that $Q^{-1}AQ = D$.

(a)

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

(e)

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}$$

(f)

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{pmatrix}$$

**Solution.** (a) We compute the characteristic polynomial:

$$\begin{aligned} f(t) &= \det\left(\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} - t\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) \\ &= \det\begin{pmatrix} 1-t & 2 \\ 0 & 1-t \end{pmatrix} \\ &= (1-t)^2 \end{aligned}$$

So, the only eigenvalue is $\lambda = 1$.

Next, we compute a basis for the eigenspace:

$$\left(\begin{array}{cc|c} 0 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

The solution to this homogeneous system of equations is the span of the vector

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Since the algebraic multiplicity of the eigenvalue is 2 and the dimension of the eigenspace is 1, this matrix is not diagonalizable.

(e)

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}$$

We compute the characteristic polynomial:

$$\begin{aligned} f(t) &= \det\left(\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix} - t\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right) \\ &= \det\begin{pmatrix} -t & 0 & 1 \\ 1 & -t & -1 \\ 0 & 1 & 1-t \end{pmatrix} \\ &= -t^3 + t^2 - t + 1 \\ &= -(t-1)(t^2+1). \end{aligned}$$

Since the characteristic polynomial doesn't split over $\mathbb{R}$, this matrix is not diagonalizable over $\mathbb{R}$.

This matrix is, however, diagonalizable over $\mathbb{C}$. The matrix has three distinct eigenvalues—1 and $\pm i$— and since eigenvectors corresponding to distinct eigenvalues are linearly independent, we see that this matrix is diagonalizable.

(f)

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{pmatrix}$$

We compute the characteristic polynomial:

$$\begin{aligned} f(t) &= \det\left(\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{pmatrix} - t\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right) \\ &= \det\begin{pmatrix} 1-t & 1 & 0 \\ 0 & 1-t & 2 \\ 0 & 0 & 3-t \end{pmatrix} \\ &= (1-t)^2(3-t) \end{aligned}$$

So, the two eigenvalues is $\lambda = 1$ and $\lambda = 3$.

Next, we compute a basis for the eigenspace for $\lambda = 1$:

$$\left(\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right)$$

The solution to this homogeneous system of equations is the span of the vector

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

Since the algebraic multiplicity of this eigenvalue is 2 and the dimension of this eigenspace is 1, this matrix is not diagonalizable.

**Problem 5.2.7.** For

$$A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix} \in M_{2\times 2}(\mathbb{R}),$$

find an expression for $A^n$, where $n$ is an arbitrary positive integer.

**Solution.** Let

$$A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}.$$

First, we diagonalize $A$.

The characteristic polynomial for $A$ is

$$\begin{aligned} \det \begin{pmatrix} 1-t & 4 \\ 2 & 3-t \end{pmatrix} &= (1-t)(3-t) - 8 \\ &= t^2 - 4t - 5 \\ &= (t-5)(t+1) \end{aligned}$$

The eigenvalues are $\lambda = -1, 5$.

Now, we find the eigenspaces.

For $\lambda = -1$, we need the solution of the system

$$\left(\begin{array}{cc|c} 2 & 4 & 0 \\ 2 & 4 & 0 \end{array}\right).$$

The solution space is the span of the vector $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$.

For $\lambda = 5$, we need the solution of the system

$$\left(\begin{array}{cc|c} -4 & 4 & 0 \\ 2 & -2 & 0 \end{array}\right).$$

The solution space is the span of the vector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Let

$$Q = \begin{pmatrix} -2 & 1 \\ 1 & 1 \end{pmatrix}.$$

Then

$$Q^{-1}AQ = \begin{pmatrix} -1 & 0 \\ 0 & 5 \end{pmatrix}.$$

Call this matrix $D$.

Then

$$
\begin{aligned}
A^n &= (QDQ^{-1})^n \\
&= QD^nQ^{-1} \\
&= \begin{pmatrix} -2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 5 \end{pmatrix}^n \begin{pmatrix} -2 & 1 \\ 1 & 1 \end{pmatrix}^{-1} \\
&= \begin{pmatrix} -2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} (-1)^n & 0 \\ 0 & 5^n \end{pmatrix} \begin{pmatrix} -\frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix} \\
&= \begin{pmatrix} \frac{2}{3}(-1)^n + \frac{1}{3}5^n & -\frac{2}{3}(-1)^n + \frac{2}{3}5^n \\ \frac{1}{3}(-1)^n + \frac{1}{3}5^n & \frac{1}{3}(-1)^n + \frac{2}{3}5^n \end{pmatrix}.
\end{aligned}
$$

**Problem 5.2.20.** Let $W_1$, $W_2$, $\ldots$, $W_k$ be subspaces of a finite-dimensional vector space $V$ such that

$$\sum_{i=1}^{k} W_i = V.$$

Prove that $V$ is the direct sum of $W_1$, $W_2$, $\ldots$, $W_k$ if and only if

$$\dim(V) = \sum_{i=1}^{k} \dim(W_i).$$

**Solution.** *Proof.* Let $W_1$, $W_2$, $\ldots$, $W_k$ be subspaces of a finite-dimensional vector space $V$ such that

$$\sum_{i=1}^{k} W_i = V.$$

We prove the result by induction on $k$, with $k = 1$ being trivial.

Suppose $k = 2$. Since $V = W_1 + W_2$, we will have $V = W_1 \oplus W_2$ if and only if $W_1 \cap W_2 = \{0\}$. By the Dimension Theorem, we have

$$\dim(V) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2).$$

From this we see that $W_1 \cap W_2 = \{0\}$ if and only if

$$\dim(V) = \dim(W_1) + \dim(W_2).$$

This proves the result for $k = 2$.

Suppose the result if true for $k = \ell$. Let $W_1$, $W_2$, $\ldots$, $W_\ell$, $W_{\ell+1}$ be subspaces of a finite-dimensional vector space $V$ such that

$$\sum_{i=1}^{\ell+1} W_i = V.$$

Let

$$W = \sum_{i=1}^{\ell} W_i.$$

By the inductive hypothesis, $W = \bigoplus_{i=1}^{\ell} W_i$ if and only if

$$\dim(W) = \sum_{i=1}^{\ell} \dim(W_i). \tag{1}$$

From the case $k = 2$ above, we know that

$$V = \bigoplus_{i=1}^{\ell+1} W_i = W \oplus W_{\ell+1}$$

if and only if $W$ is a direct sum of $W_1, \ldots, W_\ell$ and $\dim(V) = \dim(W) + \dim(W_{\ell+1})$. Combining this with Equation (1), we have

$$V = \bigoplus_{i=1}^{\ell+1} W_i = W \oplus W_{\ell+1}$$

if and only if

$$\begin{aligned}
\dim(V) &= \dim(W) + \dim(W_{\ell+1}) \\
&= \left(\sum_{i=1}^{\ell} \dim(W_i)\right) + \dim(W_{\ell+1}) \\
&= \sum_{i=1}^{\ell+1} \dim(W_i).
\end{aligned}$$

This concludes the induction and proves the result. □

**Problem 5.4.5.** Let $T$ be a linear operator on a vector space $V$. Prove that the intersection of any collection of $T$-invariant subspaces of $V$ is a $T$-invariant subspace of $V$.

**Solution.** *Proof.* Let $T$ be a linear operator on a vector space $V$ and let $\{W_\alpha\}_{\alpha \in J}$ be any collection of $T$-invariant subspaces of $V$. We consider $W = \bigcap_{\alpha \in J} W_\alpha$.

Let $v \in W$. Then $v \in W_\alpha$ for all $\alpha \in J$. Since each $W_\alpha$ is $T$-invariant, $T(v) \in W_\alpha$ for all $\alpha \in J$. Hence $T(v) \in W$. Since $v \in W$ is arbitrary, $W$ is a $T$-invariant subspaces of $V$. □

**Problem 5.4.11.** Let $T$ be a linear operator on a vector space $V$, let $v$ be a nonzero vector in $V$, and let $W$ be the $T$-cyclic subspace of $V$ generated by $v$. Prove that

(a) $W$ is $T$-invariant.

(b) Any $T$-invariant subspace of $V$ containing $v$ also contains $W$.

**Solution.** Let $T$ be a linear operator on a vector space $V$, let $v$ be a nonzero vector in $V$, and let $W$ be the $T$-cyclic subspace of $V$ generated by $v$.

(a) *Proof.* Since $W$ is the $T$-cyclic subspace of $V$ generated by $v$,

$$W = \text{span}(\{v, T(v), T^2(v), \dots\}).$$

Let $\alpha \in W$. Then

$$\alpha = \sum_{j=0}^{n} \alpha_j T^j(v),$$

by the definition of $W$. Then

$$\begin{aligned} T(\alpha) &= T(\sum_{j=0}^{n} \alpha_j T^j(v)) \\ &= \sum_{j=0}^{n} \alpha_j T(T^j(v)) \\ &= \sum_{j=0}^{n} \alpha_j T^{j+1}(v), \end{aligned}$$

which we see is back in $W$. So, $W$ is a $T$-invariant subspace of $V$. □

(b) *Proof.* Let $S$ be a $T$-invariant subspace of $V$ containing $v$. Since $S$ is $T$-invariant and $v \in S$, $T(v) \in S$. Inductively, $T^n(v) \in S$, for all $n \in \mathbb{N}$ and since $S$ is a subspace, $S$ must contain the span of this set of vectors, which is precisely $W$. So $S$ contains $W$. □

**Problem 5.4.18.** Let $A$ be an $n \times n$ matrix with characteristic polynomial

$$f(t) = (-1)^n t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0.$$

(a) Prove that $A$ is invertible if and only if $a_0 \neq 0$.

(b) Prove that if $A$ is invertible, then

$$A^{-1} = (-1/a_0)\left[(-1)^n A^{n-1} + a_{n-1}A^{n-2} + \cdots + a_1 I_n\right].$$

(c) Use (b) to compute $A^{-1}$ for

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & -1 \end{pmatrix}.$$

**Solution.** Let $A$ be an $n \times n$ matrix with characteristic polynomial

$$f(t) = (-1)^n t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0.$$

(a) The characteristic polynomial of $A$ is defined by $f(t) = \det(A - tI_n)$. Evaluating at $t = 0$, we get $f(0) = \det(A)$. Since $A$ is invertible if and only if $\det(A) \neq 0$, we see that $A$ is invertible if and only if $f(0) = a_0$ is not equal to zero.

(b) Suppose $A$ is invertible. By part (a), $a_0 \neq 0$. By the Cayley-Hamilton Theorem,

$$f(A) = (-1)^n A^n + a_{n-1}A^{n-1} + \cdots + a_1 A + a_0 I = 0.$$

By a bit of algebra, we have

$$\begin{aligned} 0 &= (-1)^n A^n + a_{n-1}A^{n-1} + \cdots + a_1 A + a_0 I \\ 0 &= A\left[(-1)^n A^{n-1} + a_{n-1}A^{n-2} + \cdots + a_1\right] + a_0 I \\ -a_0 I &= A\left[(-1)^n A^{n-1} + a_{n-1}A^{n-2} + \cdots + a_1\right] \\ I &= -\frac{1}{a_0}\left[A\left((-1)^n A^{n-1} + a_{n-1}A^{n-2} + \cdots + a_1\right)\right] \\ I &= A\left(-\frac{1}{a_0}\left[(-1)^n A^{n-1} + a_{n-1}A^{n-2} + \cdots + a_1\right]\right). \end{aligned}$$

So, we see that

$$A^{-1} = -\frac{1}{a_0}\left[(-1)^n A^{n-1} + a_{n-1}A^{n-2} + \cdots + a_1\right].$$

(c) Use (b) to compute $A^{-1}$ for

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & -1 \end{pmatrix}.$$

We first compute the characteristic polynomial for $A$.

$$\begin{aligned} f(t) &= \det(A - tI) \\ &= \det \begin{pmatrix} 1-t & 2 & 1 \\ 0 & 2-t & 3 \\ 0 & 0 & -1-t \end{pmatrix} \\ &= -t^3 + 2t^2 + t - 2. \end{aligned}$$

Since $f(0) = a_0 = -2$, $A$ is invertible and

$$A^{-1} = \frac{1}{2}(-A^2 + 2A + I) = \begin{pmatrix} 1 & -1 & -2 \\ 0 & \frac{1}{2} & \frac{3}{2} \\ 0 & 0 & -1 \end{pmatrix}.$$

**Problem 5.4.19.** Let $A$ denote the $k \times k$ matrix

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & \cdots & 0 & -a_2 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & -a_{k-2} \\ 0 & 0 & \cdots & 1 & -a_{k-1} \end{pmatrix}$$

where $a_0$, $a_1$, $\ldots$, $a_{k-1}$ are arbitrary scalars. Prove that the characteristic polynomial of $A$ is

$$(-1)^k(a_0 + a_1 t + \cdots + a_{k-1}t^{t-1} + t^k).$$

*Hint:* Use mathematical induction on $k$, expanding the determinant along the first row.

**Solution.** *Proof.* Let $A$ denote the $k \times k$ matrix

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & \cdots & 0 & -a_2 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & -a_{k-2} \\ 0 & 0 & \cdots & 1 & -a_{k-1} \end{pmatrix}$$

where $a_0$, $a_1$, $\ldots$, $a_{k-1}$ are arbitrary scalars. We prove that the characteristic polynomial of $A$ is

$$(-1)^k(a_0 + a_1 t + \cdots + a_{k-1}t^{k-1} + t^k)$$

by induction on $k$.

For $k = 1$,

$$A = \begin{pmatrix} -a_0 \end{pmatrix}$$

So, the characteristic polynomial is $-a_0 - t = -(1)^1(a_0 + t)$. So, the result is true for $k = 1$.

Assume the result is true for some $k$. That is, if

$$A = \begin{pmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & \cdots & 0 & -a_2 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & -a_{k-2} \\ 0 & 0 & \cdots & 1 & -a_{k-1} \end{pmatrix},$$

then

$$f(t) = (-1)^k(a_0 + a_1 t + \cdots + a_{k-1}t^{k-1} + t^k)$$

Consider the case $k+1$. In that case,

$$A = \begin{pmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & \cdots & 0 & -a_2 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & -a_{k-2} \\ 0 & 0 & \cdots & 1 & -a_{k-1} \\ 0 & 0 & \cdots & 1 & -a_k \end{pmatrix}.$$

The characteristic polynomial is then

$$f(t) = \det \begin{pmatrix} -t & 0 & \cdots & 0 & -a_0 \\ 1 & -t & \cdots & 0 & -a_1 \\ 0 & 1 & \cdots & 0 & -a_2 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & -a_{k-2} \\ 0 & 0 & \cdots & -t & -a_{k-1} \\ 0 & 0 & \cdots & 1 & -a_k - t \end{pmatrix}.$$

We evaluate this determinant by expanding the determinant along the first row:

$$\begin{aligned} f(t) &= \det \begin{pmatrix} -t & 0 & \cdots & 0 & -a_0 \\ 1 & -t & \cdots & 0 & -a_1 \\ 0 & 1 & \cdots & 0 & -a_2 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & -a_{k-2} \\ 0 & 0 & \cdots & -t & -a_{k-1} \\ 0 & 0 & \cdots & 1 & -a_k - t \end{pmatrix} \\ &= (-t) \det \begin{pmatrix} -t & \cdots & 0 & -a_1 \\ 1 & \cdots & 0 & -a_2 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & -a_{k-2} \\ 0 & \cdots & -t & -a_{k-1} \\ 0 & \cdots & 1 & -a_k - t \end{pmatrix} + (-1)^k(-a_0) \det \begin{pmatrix} 1 & -t & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & -t \\ 0 & 0 & \cdots & 1 \end{pmatrix} \\ &= (-t)(-1)^k(a_1 + a_2 t + \cdots + a_k t^{t-1} + t^k) + (-1)^{k+1} a_0 \\ &= (-1)^{k+1}(a_0 + a_1 t + a_2 t^2 + \cdots + a_k t^k + t^{k+1}). \end{aligned}$$

This shows the result holds for $k+1$. By the Principle of Mathematical Induction, if

$$A = \begin{pmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & \cdots & 0 & -a_2 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & -a_{k-2} \\ 0 & 0 & \cdots & 1 & -a_{k-1} \end{pmatrix}$$

where $a_0$, $a_1$, $\ldots$, $a_{k-1}$ are arbitrary scalars, then the characteristic polynomial of $A$ is

$$(-1)^k(a_0 + a_1 t + \cdots + a_{k-1}t^{t-1} + t^k).$$

□