Problem Set #11 Solutions
Due Thursday, October 30

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Problem 4.4.3. Evaluate the determinant of the following matrices in the manner indicated.

(e)

$$\begin{pmatrix} 0 & 1+i & 2 \\ -2i & 0 & 1-i \\ 3 & 4i & 0 \end{pmatrix}$$

along the third row

Solution. (e)

$$\begin{aligned} \det \begin{pmatrix} 0 & 1+i & 2 \\ -2i & 0 & 1-i \\ 3 & 4i & 0 \end{pmatrix} &= (3) \det \begin{pmatrix} 1+i & 2 \\ 0 & 1-i \end{pmatrix} - (4i) \det \begin{pmatrix} 0 & 2 \\ -2i & 1-i \end{pmatrix} \\ &= (3)(2) - (4i)(4i) \\ &= 22. \end{aligned}$$

Problem 4.4.4. Evaluate the determinant of the following matrices by any legitimate method.

(f) $\begin{pmatrix} -1 & 2+i & 3 \\ 1-i & i & 1 \\ 3i & 2 & -1+i \end{pmatrix}$

Solution. (f) We start with the matrix $M = \begin{pmatrix} -1 & 2+i & 3 \\ 1-i & i & 1 \\ 3i & 2 & -1+i \end{pmatrix}$.

We will compute the determinant by expansion by minors along the first row.

We have

$$\begin{aligned} \det(M) &= (-1)\begin{vmatrix} i & 1 \\ 2 & -1+i \end{vmatrix} - (2+i)\begin{vmatrix} 1-i & 1 \\ 3i & -1+i \end{vmatrix} + 3\begin{vmatrix} 1-i & i \\ 3i & 2 \end{vmatrix} \\ &= (-1)[i(-1+i) - (1)(2)] - (2+i)[(1-i)(-1+i) - (3i)(1)] + 3[2(1-i) - (3i)(i)] \\ &= (-1)(-i-3) - (2+i)(-i) + 3(5-2i) \\ &= (3+i) + (-1+2i) + (15-6i) \\ &= 17-3i. \end{aligned}$$

Problem 4.4.6. Suppose that $M \in M_{n\times n}(\mathbb{F})$ can be written in the form

$$M = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix},$$

where A and C are square matrices, then $\det(M) = \det(A) \cdot \det(C)$.

Solution. Suppose that $M \in M_{n\times n}(\mathbb{F})$ can be written in the form

$$M = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix},$$

where A and C are square matrices. Suppose $C \in M_{k\times k}(\mathbb{F})$. Let $Q \in M_{k\times k}(\mathbb{F})$ be an invertible matrix so that $QC = R$, where R is the reduced row echelon form of C.

Let $N \in M_{n\times n}(\mathbb{F})$ be defined by

$$N = \begin{pmatrix} I & 0 \\ 0 & Q \end{pmatrix}.$$

We remark that $\det(N) = \det(Q)$ (essentially) by the last problem.

We note that

$$NM = \begin{pmatrix} I & 0 \\ 0 & Q \end{pmatrix}\begin{pmatrix} A & B \\ 0 & C \end{pmatrix} = \begin{pmatrix} A & B \\ 0 & QC \end{pmatrix} = \begin{pmatrix} A & B \\ 0 & R \end{pmatrix}$$

Now, we note that R is an $k \times k$ matrix in reduced row echelon form. If $\operatorname{rank}(R) = k$, then R is the identity matrix $\det(R) = 1$. If $\operatorname{rank}(R) < k$, then R is singular and $\det(R) = 0$.

Suppose R is singular. Then $\det(R) = 0$. Since $QC = R$, we have $\det(R) = \det(QC) = \det(Q)\det(C)$. Since $\det(R) = 0$ and, since Q is invertible, $\det(Q) \neq 0$, we see that $\det(C) = 0$.

On the other hand, since $NM = \begin{pmatrix} A & B \\ 0 & R \end{pmatrix}$, by the first part of this problem, $\det(NM) = \det(A)\det(R) = 0$. Also, $\det(NM) = \det(N)\det(M)$. Since N is invertible and therefore $\det(N) \neq 0$, this implies that $\det(M) = 0$. So, $\det(M) = \det(A) \cdot \det(C) = 0$.

Now suppose R is invertible. As remarked above, $\det(R) = 1$.

By the first part of this problem,

$$\det(NM) = \det(A)\det(R) = \det(A)\,. \tag{1}$$

Also by the first part of this problem, we have (essentially)

$$\det(N) = \det(Q)\,. \tag{2}$$

Since $R = QC$, we have $1 = \det(R) = \det(Q)\det(C)$. So,

$$\det(C) = [\det(Q)]^{-1}. \tag{3}$$

So,

$$\begin{aligned}\det(A)\det(C) &= \det(A)\,[\det(Q)]^{-1} \text{ by Equation (3)}\\ &= \det(NM)\,[\det(Q)]^{-1} \text{ by Equation (1)}\\ &= \det(NM)\,[\det(N)]^{-1} \text{ by Equation (2)}\\ &= \det(M)\det(N)\,[\det(N)]^{-1}\\ &= \det(M)\,.\end{aligned}$$

Problem 5.1.2. For each of the following linear operators T on a vector space V and ordered bases β, compute $[T]_\beta$, and determine whether β is a basis consisting of eigenvectors of T.

(e) $V = P_3(\mathbb{R})$,

$$T(a + bx + cx^2 + dx^3) = -d + (-c + d)x + (a + b - 2c)x^2 + (-b + c - 2d)x^3,$$

and $\beta = \{1 - x + x^3, 1 + x^2, 1, x + x^2\}$.

Solution. (e) Let $V = P_3(\mathbb{R})$ with basis $\beta = \{1 - x + x^3, 1 + x^2, 1, x + x^2\}$. Let

$$T(a + bx + cx^2 + dx^3) = -d + (-c + d)x + (a + b - 2c)x^2 + (-b + c - 2d)x^3,$$

Then

$$\begin{aligned} T(1 - x + x^3) &= -1 + x - x^3 \\ T(1 + x^2) &= -x - x^2 + x^3 \\ T(1) &= x^2 \\ T(x + x^2) &= -x - x^2 \end{aligned}$$

We see the first and fourth polynomials are eigenvectors, but the second and third are not.

We compute the matrix for T with respect to the standard basis $\sigma = \{1, x, x^2, x^3\}$:

$$[T]_\sigma = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 1 \\ 1 & 1 & -2 & 0 \\ 0 & -1 & 1 & -2 \end{pmatrix}$$

The change of basis matrix is

$$Q = [I]_\beta^\sigma = \begin{pmatrix} 1 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

Then, by the change of basis formula

$$\begin{aligned} [T]_\beta &= Q^{-1}[T]_\sigma Q \\ &= \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \end{aligned}$$

Problem 5.1.7. Let T be a linear operator on a finite-dimensional vector space V. We define the **determinant** of T, denoted $\det(T)$, as follows: Choose any ordered basis β for V, and define $\det(T) = \det([T]_\beta)$.

(a) Prove that the preceding definition is independent of the choice of an ordered basis for V. That is, prove that if β and γ are two ordered bases for V, then $\det([T]_\beta) = \det([T]_\gamma)$.

(b) Prove that T is invertible if and only if $\det(T) \neq 0$.

(c) Prove that if T is invertible, then $\det\left(T^{-1}\right) = [\det(T)]^{-1}$.

(d) Prove that if U is also a linear operator on V, then $\det(TU) = \det(T) \cdot \det(U)$.

(e) Prove that $\det(T - \lambda I_V) = \det([T]_\beta - \lambda I)$ for any scalar λ and any ordered basis β for V.

Solution. (a) We know that

$$[T]_\beta = [I]_\gamma^\beta [T]_\gamma [I]_\beta^\gamma$$

and

$$\left([I]_\gamma^\beta\right)^{-1} = [I]_\beta^\gamma$$

So, we have that

$$\begin{aligned}
\det([T]_\beta) &= \det\left([I]_\gamma^\beta [T]_\gamma [I]_\beta^\gamma\right) \\
&= \det\left([I]_\gamma^\beta\right) \det([T]_\gamma) \det\left([I]_\beta^\gamma\right) \\
&= \det\left([I]_\gamma^\beta\right) \det([T]_\gamma) \det\left(\left([I]_\gamma^\beta\right)^{-1}\right) \\
&= \det\left([I]_\gamma^\beta\right) \det([T]_\gamma) \left(\det\left([I]_\gamma^\beta\right)\right)^{-1} \\
&= \det([T]_\gamma).
\end{aligned}$$

(b) The transformation T is invertible if and only if $[T]_\beta$ is invertible for any basis β. We know that $[T]_\beta$ is invertible if and only if $\det([T]_\beta) \neq 0$. Since $\det(T) = \det([T]_\beta)$, the transformation T is invertible is and only if $\det(T) \neq 0$.

(c) We have that

$$\det\left(T^{-1}\right) = \det\left([T^{-1}]_\beta\right) = \det\left([T]_\beta^{-1}\right) = [\det([T]_\beta)]^{-1} = [\det(T)]^{-1}$$

(d) We have that

$$\det(TU) = \det([TU]_\beta) = \det([T]_\beta [U]_\beta) = \det([T]_\beta) \det([U]_\beta) = \det(T) \det(U).$$

(e) We have that

$$\det(T - \lambda I_V) = \det([T - \lambda I_V]_\beta) = \det([T]_\beta - \lambda [I_V]_\beta) = \det([T]_\beta - \lambda I)$$

Problem 5.1.20. Let A be an $n \times n$ matrix with characteristic polynomial

$$f(t) = (-1)^n t^n + a_{n-1} t^{n-1} + \cdots + a_1 t + a_0.$$

Prove that $f(0) = a_0 = \det(A)$. Deduce that A is invertible if and only if $a_0 \neq 0$.

Solution. Let A be an $n \times n$ matrix with characteristic polynomial

$$f(t) = (-1)^n t^n + a_{n-1} t^{n-1} + \cdots + a_1 t + a_0.$$

By definition, the characteristic polynomial for A is

$$f(t) = \det\bigl(A - tI\bigr) .$$

By setting $t = 0$, we see that

$$f(0) = a_0 = \det(A) .$$