Problem Set #10 Solutions
Due Thursday, October 23

William M. Faucette

Problem 4.2.23. Prove that the determinant of an upper triangular matrix is the product of its diagonal entries.

Solution. *Proof.* Let A be an upper triangular $n \times n$ matrix. We proceed by induction on n. For $n = 1$, there's nothing to prove. For $n = 2$, we have

$$\det \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} = ad - b \cdot 0 = ad.$$

So, the result is true for $n = 2$.

Assume the result is true for some $k \in \mathbb{N}$.

Let A be an upper triangular $(k+1) \times (k+1)$ matrix. Let A' be the $k \times k$ matrix obtained by deleting the first row and first column from A. Since A' is also an upper triangular matrix with the same diagonal entries as A for rows (and columns) 2 through $k+1$, by the inductive hypothesis, $\det(A') = a_{22}a_{33}a_{44} \cdots a_{k+1,k+1}$.

If we expand $\det(A)$ along the first column of A, there is only one nonzero entry:

$$\begin{aligned} \det(A) &= a_{11} \det\bigl(A'\bigr) \\ &= a_{11} \left[a_{22}a_{33}a_{44} \cdots a_{k+1,k+1}\right] \\ &= a_{11}a_{22} \cdots a_{k+1,k+1}. \end{aligned}$$

So, we see the result is true for $k + 1$.

By the Principle of Mathematical Induction, the determinant of any upper triangular matrix is the product of its diagonal entries. □

Problem 4.2.25. Prove that $\det(kA) = k^n \det(A)$ for any $A \in M_{n\times n}(\mathbb{F})$.

Solution 1. *Proof.* Let $A \in M_{n\times n}(\mathbb{F})$. We prove $\det(kA) = k^n \det(A)$ by induction on n.

For $n = 1$, there's nothing to prove.

Suppose the result is true for some $m \in \mathbb{N}$. That is, assume that

$$\det(kA) = k^m \det(A)$$

whenever $A \in M_{m\times m}(\mathbb{F})$.

Now, let $A = [a_{ij}] \in M_{m+1\times m+1}(\mathbb{F})$. Let A_{ij} denote the $m \times m$ matrix obtained from A by deleting the ith row and the jth column. Let $B = kA$. Notice if $B = [b_{ij}]$ then $b_{ij} = ka_{ij}$. Let B_{ij} denote the $m \times m$ matrix obtained from B by deleting the ith row and the jth column.

Noting that $B_{ij} = kA_{ij}$, by the induction hypothesis, we have $\det(B_{ij}) = k^m \det(A_{ij})$.

Let's compute $\det(A)$ by expansion along the last row:

$$\begin{aligned}
\det(B) &= \sum_{i=1}^{m+1} (-1)^{i+m+1} b_{m+1,i} \det(B_{m+1,i}) \\
&= \sum_{i=1}^{m+1} (-1)^{i+m+1} k a_{m+1,i} \cdot k^m \det(A_{m+1,i}) \\
&= k^{m+1} \sum_{i=1}^{m+1} (-1)^{i+m+1} a_{m+1,i} \det(A_{m+1,i}) \\
&= k^{m+1} \det(A)\,.
\end{aligned}$$

This conclude the induction and the proof. □

Solution 2. *Proof.* Let $A \in M_{n\times n}(\mathbb{F})$. Then

$$kA = k[(I_n)A] = [k(I_n)]A,$$

so that

$$\det(kA) = \det([k[(I_n)]A) = \det([k(I_n)]) \det(A)\,.$$

Now, $k(I_n)$ is an upper triangular matrix with k's along the diagonal, so by Exercise 4.2 #23, $\det(k(I_n)) = k^n$. Thus, we see that

$$\det(kA) = \det([k[(I_n)]) \det(A) = k^n \det(A)\,,$$

as desired. □

Problem 4.2.26. Let $A \in M_{n\times n}(\mathbb{F})$. Under what conditions is $\det(-A) = \det(A)$?

Solution. Let $A \in M_{n\times n}(\mathbb{F})$. From Exercise 4.2 #25, we know that $\det(-A) = (-1)^n \det(A)$. So, $\det(-A) = \det(A)$ if n is even. If n is odd, we have $\det(-A) = -\det(A)$. From this, we see that $\det(-A) = \det(A)$ if and only if $\det(A) = -\det(A)$, and this occurs if and only if $\det(A) = 0$ or $\mathbb{F}$ has characteristic 2.

Problem 4.2.27. Prove that if $A \in M_{n\times n}(\mathbb{F})$ has two identical columns, then $\det(A) = 0$.

Solution. *Proof.* Suppose $A \in M_{n\times n}(\mathbb{F})$ has two identical columns. Say columns i and j are equal.

Perform the elementary column operation of adding -1 times column i to column j. This elementary column does not change the determinant, but the resulting matrix has a column of zeroes, and so has determinant zero.

We see then that $\det(A) = 0$. □

Problem 4.3.9. Prove that an upper triangular $n \times n$ matrix is invertible if and only if all its diagonal entries are nonzero.

Solution. Any square matrix is invertible if and only if its determinant is nonzero. The determinant of an upper triangular square matrix is the product of its diagonal elements. Putting these together, an upper triangular matrix is invertible if and only if all its diagonal entries are nonzero.

Problem 4.3.10. A matrix $M \in M_{n\times n}(\mathbb{C})$ is called **nilpotent** if, for some positive integer k, $M^k = 0$, where 0 is the $n\times n$ zero matrix. Prove that if M is nilpotent, then $\det(M) = 0$.

Solution. Let M be a nilpotent $n \times n$ matrix. So, there exists $k \in \mathbb{N}$ so that $M^k = 0$. Then

$$0 = \det(0) = \det\left(M^k\right) = [\det(M)]^k,$$

so it follows that $\det(M) = 0$.

Problem 4.3.11. A matrix $M \in M_{n\times n}(\mathbb{C})$ is called **skew-symmetric** if $M^t = -M$. Prove that if M is **skew-symmetric** and n is odd, then M is not invertible. What happens if n is even?

Solution. Let $M \in M_{n\times n}(\mathbb{C})$ be a skew-symmetric matrix with n odd. Then

$$\det(M) = \det\big(M^t\big) = \det(-M) = (-1)^n \det(M) = -\det(M)\,.$$

Consequently, $\det(M) = 0$.

As for what happens when n is even, the matrix $\begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}$ is skew-symmetric, but it has determinant 4.

Problem 4.3.12. A matrix $Q \in M_{n\times n}(\mathbb{C})$ is called **orthogonal** if $QQ^t = I$. Prove that if Q is orthogonal, then $\det(Q) = \pm 1$.

Solution. Let $Q \in M_{n\times n}(\mathbb{C})$ be an orthogonal matrix. Then

$$1 = \det(I) = \det\bigl(QQ^t\bigr) = \det(Q)\det\bigl(Q^t\bigr) = \det(Q)\det(Q) = [\det(Q)]^2$$

Consequently, $\det(Q) = \pm 1$.

Problem 4.3.15. Prove that if A, $B \in M_{n\times n}(\mathbb{F})$ are similar, then $\det(A) = \det(B)$.

Solution. Let A, $B \in M_{n\times n}(\mathbb{F})$ be similar. Then there exists an invertible matrix $P \in M_{n\times n}(\mathbb{F})$ so that $B = PAP^{-1}$. But then

$$\det(B) = \det\bigl(PAP^{-1}\bigr) = \det(P)\det(A)\,[\det(P)]^{-1} = \det(A)\,.$$