Problem Set #6
Due Thursday, September 25

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Problem 2.6.1. Label the following statements as true or false. Assume that all vector spaces are finite-dimensional.

(a) Every linear transformation is a linear functional.

(b) A linear functional defined on a field may be represented as a 1×1 matrix.

(c) Every vector space is isomorphic to its dual space.

(d) Every vector space is the dual of some other vector space.

(e) If T is an isomorphism from V into V^* and β is a finite ordered basis for V, then $T(\beta) = \beta^*$.

(f) If T is a linear transformation from V to W, then the domain of $(T^t)^t$ is $(V^*)^*$.

(g) If V is isomorphic to W, then V^* is isomorphic to W^*.

(h) The derivative of a function may be considered as a linear functional on the vector space of differentiable functions.

Problem 2.6.4. Let $V = \mathbb{R}^3$, and define $f_1,\ f_2,\ f_3 \in V^*$ as follows:

$$f_1(x, y, z) = x - 2y, \quad f_2(x, y, z) = x + y + z, \quad f_3(x, y, z) = y - 3z.$$

Prove that $\{f_1, f_2, f_3\}$ is a basis for V^*, and then find a basis for V for which it is the dual basis.

Problem 2.6.5. Let $V = P_1(\mathbb{R})$, and, for $p(x) \in V$, define $f_1,\ f_2 \in V^*$ by

$$f_1(p(x)) = \int_0^1 p(t)\,dt \text{ and } f_2(p(x)) = \int_0^2 p(t)\,dt.$$

Prove that $\{f_1, f_2\}$ is a basis for V^*, and find a basis for V for which it is the dual basis.

Problem 2.6.8. Show that every plane through the origin in $\mathbb{R}^3$ may be identified with the null space of a vector in $(\mathbb{R}^3)^*$. State an analogous result for $\mathbb{R}^2$.

Problem 3.1.1. Label the following statements as true or false.

(a) An elementary matrix is always square.

(b) The only entries of an elementary matrix are zeros and ones.

(c) The $n \times n$ identity matrix is an elementary matrix.

(d) The product of two $n \times n$ elementary matrices is an elementary matrix.

(e) The inverse of an elementary matrix is an elementary matrix.

(f) The sum of two $n \times n$ elementary matrices is an elementary matrix.

(g) The transpose of an elementary matrix is an elementary matrix.

(h) If B is a matrix that can be obtained by performing an elementary row operation on a matrix A, then B can also be obtained by performing an elementary column operation on A.

(i) If B is a matrix that can be obtained by performing an elementary row operation on a matrix A, then A can be obtained by performing an elementary row operation on B.

Problem 3.1.2. Let

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ 1 & -1 & 1 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 & 3 \\ 1 & -2 & 1 \\ 1 & -3 & 1 \end{pmatrix}, C = \begin{pmatrix} 1 & 0 & 3 \\ 0 & -2 & -2 \\ 1 & -3 & 1 \end{pmatrix}.$$

Find an elementary operation that transforms A into B and an elementary operation that transforms B into C. By means of several additional operations, transform C into I_3.

Problem 3.1.3. Use the proof of Theorem 3.2 to obtain the inverse of each of the following elementary matrices.

(a) $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

(b) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

(c) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}$

Problem 3.1.5. Prove that E is an elementary matrix if and only if E^t is.