Problem Set #4
Due Thursday, September 11

William M. Faucette

Problem 2.2.4. Define

$$T : M_{2\times 2}(\mathbb{R}) \to P_2(\mathbb{R}) \text{ by } T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = (a+b) + (2d)x + bx^2.$$

Let

$$\beta = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}, \text{ and } \gamma = \{1, x, x^2\}.$$

Compute $[T]_\beta^\gamma$.

Problem 2.2.8. Let V be an n-dimensional vector space with an ordered basis β. Define $T : V \to \mathbb{F}^n$ by $T(x) = [x]_\beta$. Prove that T is linear.

Problem 2.2.9. Let V be the vector space of complex numbers over the field $\mathbb{R}$. Define $T : V \to V$ by $T(z) = \overline{z}$, where $\overline{z}$ is the complex conjugate of z. Prove that T is linear, and compute $[T]_\beta$, where $\beta = \{1, i\}$. (Recall by Exercise 38 of Section 2.1 that T is not linear if V is regarded as a vector space over the field $\mathbb{C}$.)

Problem 2.2.15. Let V and W be vector spaces, and let S be a subset of V. Define $S^0 = \{T \in \mathcal{L}(V, W) : T(x) = 0 \text{ for all } x \in S\}$. Prove the following statements.

(a) S_0 is a subspace of $\mathcal{L}(V, W)$.

(b) If S_1 and S_2 are subsets of V and $S_1 \subseteq S_2$, then $S_2^0 \subseteq S_1^0$.

(c) If V_1 and V_2 are subspaces of V, then $(V_1 + V_2)^0 = V_1^0 \cap V_2^0$

Problem 2.3.3. Let $g(x) = 3 + x$. Let $T : P_2(\mathbb{R}) \to P_2(\mathbb{R})$ and $U : P_2(\mathbb{R}) \to \mathbb{R}^3$ be the linear transformations respectively defined by

$$T(f(x)) = f'(x)g(x) + 2f(x) \text{ and } U(a + bx + cx^2) = (a + b, c, a - b).$$

Let β and γ be the standard ordered bases of $P_2(\mathbb{R})$ and $\mathbb{R}^3$, respectively.

(a) Compute $[U]_\beta^\gamma$, $[T]_\beta$, and $[UT]_\beta^\gamma$ directly. Then use Theorem 2.11 to verify your result.

(b) Let $h(x) = 3 - 2x + x^2$. Compute $[h(x)]_\beta$ and $[U(h(x))]_\gamma$. Then use $[U]_\beta^\gamma$ from (a) and Theorem 2.14 to verify your result.

Problem 2.3.9. Find linear transformations U, $T : \mathbb{F}^2 \to \mathbb{F}^2$ such that $UT = T_0$ (the zero transformation) but $TU \neq T_0$. Use your answer to find matrices A and B such that $AB = 0$ but $BA \neq 0$.

Problem 2.3.11. Let V be a vector space, and let $T : V \to W$ be linear. Prove that $T^2 = T_0$ if and only if $R(T) \subseteq N(T)$.

Problem 2.3.13. Let A and B be $n \times n$ matrices. Recall that the trace of A is defined by

$$\mathrm{tr}(A) = \sum_{i=1}^{n} A_{ii}.$$

Prove that $\mathrm{tr}(AB) = \mathrm{tr}(BA)$ and $\mathrm{tr}(A) = \mathrm{tr}(A^t)$.