Problem Set #3
Due Thursday, September 4

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Problem 1.6.14. Find bases for the following subspaces of $\mathbb{F}^5$:

$$W_1 = \{(a_1, a_2, a_3, a_4, a_5) \in \mathbb{F}^5 : a_1 - a_3 - a_4 = 0\}$$

and

$$W_2 = \{(a_1, a_2, a_3, a_4, a_5) \in \mathbb{F}^5 : a_2 = a_3 = a_4 \text{ and } a_1 + a_5 = 0\}.$$

What are the dimensions of W_1 and W_2?

Problem 1.6.15. The set of all $n \times n$ matrices having trace equal to zero is a subspace W of $M_{n\times n}(\mathbb{F})$. Find a basis for W. What is the dimension of W?

Problem 1.6.16. The set of all upper triangular $n \times n$ matrices is a subspace W of $M_{n\times n}(\mathbb{F})$. Find a basis for W. What is the dimension of W?

Problem 1.6.17. The set of all skew-symmetric $n \times n$ matrices is a subspace W of $M_{n\times n}(\mathbb{F})$. Find a basis for W. What is the dimension of W?

Problem 2.1.5. Prove that

$$T : P_2(\mathbb{R}) \to P_3(\mathbb{R})$$
$$T(f(x)) = xf(x) + f'(x).$$

is a linear transformation, and find bases for both $N(T)$ and $R(T)$. Then compute the nullity and rank of T, and verify the dimension theorem. Finally, use the appropriate theorems in this section to determine whether T is one-to-one or onto.

Problem 2.1.6. Prove that

$$T : M_{n\times n}(\mathbb{F}) \to \mathbb{F}$$
$$T(A) = \operatorname{tr}(A) = \sum_{i=1}^{n} A_{ii}.$$

is a linear transformation, and find bases for both $N(T)$ and $R(T)$. Then compute the nullity and rank of T, and verify the dimension theorem. Finally, use the appropriate theorems in this section to determine whether T is one-to-one or onto.

Problem 2.1.14. Let V and W be vector spaces and $T : V \to W$ be linear.

(a) Prove that T is one-to-one if and only if T carries linearly independent subsets of V onto linearly independent subsets of W.

(b) Suppose that T is one-to-one and that S is a subset of V. Prove that S is linearly independent if and only if $T(S)$ is linearly independent.

(c) Suppose $\beta = \{v_1, v_2, \ldots, v_n\}$ is a basis for V and T is one-to-one and onto. Prove that $T(\beta) = \{T(v_1), T(v_2), \ldots, T(v_n)\}$ is a basis for W.