Problem Set #14
Due Thursday, November 20

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Problem 6.3.2. For each of the following inner product spaces V (over $\mathbb{F}$) and linear transformations $g : V \to \mathbb{F}$, find a vector y such that $g(x) = \langle x, y \rangle$ for all $x \in V$.

(a) $V = \mathbb{R}^3$, $g(a_1, a_2, a_3) = a_1 - 2a_2 + 4a_3$

Problem 6.3.6. Let T be a linear operator on an inner product space V. Let $U_1 = T + T^*$ and $U_2 = TT^*$. Prove that $U_1 = U_1^*$ and $U_2 = U_2^*$.

Problem 6.3.9. Prove that if $V = W \oplus W^\perp$ and T is projection on W along $W^\perp$, then $T = T^*$. *Hint:* Recall that $N(T) = W^\perp$. (For definitions, see the exercises of Sections 1.3 and 2.1.)

Problem 6.3.11. For a linear operator T on an inner product space V, prove that $T^*T = T_0$ implies $T = T_0$. Is the same result true if we assume that $TT^* = T_0$?

Problem 6.3.12. Let V be an inner product space, and let T be a linear operator on V. Prove the following results.

(a) $R(T^*)^\perp = N(T)$.

(b) If V is finite-dimensional, then $R(T^*) = N(T)^\perp$. *Hint:* Use Exercise 13(c) of Section 6.2.

Problem 6.3.18. Let A be an $n \times n$ matrix. Prove that $\det(A^*) = \overline{\det(A)}$.

Problem 6.4.3. Give an example of a linear operator T on $\mathbb{R}^2$ and an ordered basis for $\mathbb{R}^2$ that provides a counterexample to the statement in Exercise 1(c).

Problem 6.4.4. Let T and U be self-adjoint operators on an inner product space V. Prove that TU is self-adjoint if and only if $TU = UT$.

Problem 6.4.6. Let V be a complex inner product space, and let T be a linear operator on V. Define

$$T_1 = \frac{1}{2}(T + T^*) \text{ and } T_2 = \frac{1}{2i}(T - T^*).$$

(a) Prove that T_1 and T_2 are self-adjoint and that $T = T_1 + iT_2$.

(b) Suppose also that $T = U_1 + iU_2$, where U_1 and U_2 are self-adjoint. Prove that $U_1 = T_1$ and $U_2 = T_2$.

(c) Prove that T is normal if and only if $T_1T_2 = T_2T_1$.

Problem 6.4.7. Let T be a linear operator on an inner product space V, and let W be a T-invariant subspace of V. Prove that following results.

(a) If T is self-adjoint, then T_W is self-adjoint.

(b) $W^\perp$ is T^*-invariant.

(c) If W is both T- and T^*-invariant, then $(T_W)^* = (T^*)_W$.

(d) If W is both T- and T^*-invariant and T is normal, then T_W is normal.

Problem 6.4.8. Let T be a normal operator on a finite-dimensional complex inner product space V, and let W be a subspace of V. Prove that if W is T-invariant, then W is also T^*-invariant. *Hint:* Use Exercise 24 of section 5.4.