Test #2
MATH 3243

Name (printed):________________________________

Problem	**Score**
1	
2	
3	
4	
Total	

Problem 1. (25 pts) Mark each as true or false. Briefly justify your answer.

(a) Every bounded, infinite set A has a limit point.

(b) If $f : U \to \mathbb{R}$ is continuous and U is open then $f(U)$ is open.

(c) Let $A \subseteq \mathbb{R}$ and let S be the set of isolated points of A. Then S is closed.

(d) If $f : [a, b] \to \mathbb{R}$ is continuous and $f(a) \leq L \leq f(b)$, then there exists $c \in (a, b)$ such that $f(c) = L$.

(e) If a set $A \subseteq R$ has maximum and a minimum, then A is compact.

Solution. (a) This statement is true. Since A is infinite, there is a sequence (a_n) with $a_n \in A$ with all a_n distinct. The sequence is bounded, so by the Bolzano-Weierstrass Theorem, is has a convergent subsequence. The limit of this subsequence is a limit point of A.

(b) This statement is false. What is true is $f^{-1}(U)$ is open whenever U is open. A counterexample is $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x/(1 + x^2)$ which has image $[0, 1/2]$.

(c) This statement is true. Every point of S is an isolated point, so S has no limit points. So S is closed.

(d) This statement is false. We need a strict inequality $f(a) < L < f(b)$ in order to ensure that $c \in (a, b)$. A counterexample is $f(x) = x^2$ which has $f(-1) = f(1) = 1$ but $f(c) \neq 1$ for all $c \in (-1, 1)$.

(e) This statement is false. For example, $A = [0, 1) \cup (1, 2]$ is not compact since it is not closed, but $\max A = 2$ and $\min A = 0$.

Problem 2. (25 pts) Let

$$\begin{aligned} f :&\mathbb{R}_{\geq 0} \to \mathbb{R} \\ f(x) &= \sqrt{x}, \end{aligned}$$

where $R_{\geq 0} = \{x \in \mathbb{R} : x \geq 0\}$.

(a) Prove that f is continuous, using the definition.

(b) Prove that f is uniformly continuous on $[0, 1]$.

(c) Is the derivative f' of f uniformly continuous on its domain? Justify your answer rigorously.

Solution. (a) First, suppose $c = 0$. Let $\epsilon > 0$. Let $\delta = \epsilon^2$. Then, if $|x| < \delta$ then $|\sqrt{x}| < \epsilon$

Now assume $c > 0$. Let $\epsilon > 0$. Let $\delta = \epsilon\sqrt{c} > 0$. Then if $|x - c| < \delta$, we have

$$|\sqrt{x} - \sqrt{c}| = \frac{|x - c|}{|\sqrt{x} + \sqrt{c}|} = \frac{|x - c|}{\sqrt{x} + \sqrt{c}} \leq \frac{|x - c|}{\sqrt{c}} < \frac{\epsilon\sqrt{c}}{\sqrt{c}} = \epsilon.$$

(b) Since f is continuous on $[0, 1]$ which is compact, f is uniformly continuous by Theorem 4.4.7.

(c) We have $f'(x) = 1/(2\sqrt{x})$ defined on $\mathbb{R}_{>0}$. Consider the sequence $(x_n), (y_n)$ were $x_n = 1/n^2$ and $y_n = 1/(n + 1)^2$. Then $x_n, y_n \to 0$ so $|x_n - y_n| \to 0$, but $|f'(x_n) - f'(y_n)| = |n/2 - (n + 1)/2| = 1/2$. So, f' is *not* uniformly continuous on $\mathbb{R}_{>0}$ by the sequential criterion.

Problem 3. (25 pts) Let $f : A \to \mathbb{R}$ be a function and let c be a limit point of A. Suppose that

$$\lim_{x \to c} f(x) = L > 0.$$

Prove that there exists a neighborhood $U \subseteq \mathbb{R}$ of c such that $f(x) > 0$ for all $x \in U \cap A$.

Solution. Let $\epsilon = L > 0$. Since $\lim_{x \to c} f(x) = L$ exists, there exists $\delta > 0$ such that $|f(x) - L| < L$ whenever $0 < |x - c| < \delta$ and $x \in A$. Let $U = V_\delta(c)$. Then, if $x \in U \cap A$ then $|f(x) - L| < L$ so $-L < f(x) - L < L$ and thus $0 < f(x)$ as desired.

Problem 4. (25 pts) For $a \geq 0$, define $f_a : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} x^a \sin(\ln(x)), & \text{if } x > 0; \\ 0, & \text{if } x \leq 0; \end{cases}$$

(a) Compute $f'(x)$ for $x \neq 0$. (You may use the familiar rules for differentiation.)

(b) Find a value $a \in \mathbb{N}$ such that f has a continuous derivative at $c = 0$, and prove rigorously that your answer is correct.

Solution. (a) We compute for $x > 0$ that

$$f'(x) = ax^{a-1} \sin(\ln(x)) + x^{a-1} \cos(\ln(x)) = x^{a-1} \left(a \sin(\ln(x)) + \cos(\ln(x))\right)$$

and $f'(x) = 0$ if $x < 0$.

(b) We claim that f is differentiable if (and only if) $a > 1$, in which case $f'(0) = 0$. We prove this for $a = 2$. We compute

$$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{f(x)}{x}.$$

Let $\epsilon > 0$. Let $\delta = \epsilon$. Then if $x \neq 0$ and $|x| < \delta$ then

$$\left| \frac{f(x)}{x} \right| = |x \sin(\ln(x))| \leq |x| < \epsilon.$$

In fact, for $a = 2$ the derivative f' is also continuous. We prove that $\lim_{x \to 0} f'(x) = 0$. Let $\epsilon > 0$, and let $\delta = \epsilon/3$. We claim if $|x| < \delta$ then $|f'(x) - f'(0)| < \epsilon$. If $x < 0$ then $f'(x) = 0$ already; if $x > 0$ then

$$|f'(x)| = |x| \, |2 \sin(\ln(x)) + \cos(\ln(x))| \leq 3|x| < 3 \left(\frac{\epsilon}{3}\right) = \epsilon.$$