Test #1
MATH 3243

Name (printed):________________________________

Problem	**Score**
1	
2	
3	
4	
5	
Total	

Problem 1. (20 pts) Mark each as true or false. Briefly justify your answer.

(a) Every Cauchy sequence is bounded.

Statement (a) is true: A Cauchy sequence is convergent and a convergent sequence is bounded. (One can also show directly that a Cauchy sequence is bounded.)

(b) Let $A \subset \mathbb{R}$ be bounded below and let $B = \{b \in \mathbb{R} : b \text{ is a lower bound for } A\}$. Then B has a maximum.

Statement (b) is true: we have $\max B = \inf A$.

(c) The set $\{x \in \mathbb{N} : x < 17\}$ is countable.

Statement (c) is false: By your author's definition, countable sets are infinite. This set is finite.

(d) If $\sum x_n^2$ converges and $x_n \geq 0$ for all n, then $\sum x_n$ converges.

Statement (d) is false: The harmonic series $\sum 1/n$ diverges but the sequence $\sum 1/n^2$ converges.

(e) Given $a, b \in \mathbb{R}$ with $a < b$, the set of irrationals in the interval (a, b) is uncountable.

Statement (e) is true: The rationals are countable but the interval (a, b) is uncountable, so there are uncountably many irrationals left.

Problem 2. (20 pts) Prove using the definition that the sequence

$$\left(\frac{\sqrt{n}}{\sqrt{n}+1}\right)$$

converges to 1.

Solution. *Proof.* Let $\epsilon > 0$. Let $N \in \mathbb{N}$ satisfy $N > 1/\epsilon^2$. Then for $n \geq N$, we have $n \geq N > 1/\epsilon^2$ so $1/\sqrt{n} < \epsilon$ hence

$$\left|\frac{\sqrt{n}}{\sqrt{n}+1} - 1\right| = \left|\frac{\sqrt{n} - (\sqrt{n}+1)}{\sqrt{n}+1}\right| = \frac{1}{\sqrt{n}+1} < \frac{1}{\sqrt{n}} < \epsilon.$$

thus $\lim \sqrt{n}/(\sqrt{n}+1) = 1$. □

Problem 3. (20 pts) Let $A \subseteq \mathbb{R}$ be nonempty and bounded above. Suppose $x \geq 0$ for all $x \in A$. Let

$$\frac{1}{2}A = \left\{ \frac{1}{2}x : x \in A \right\}.$$

(a) Show that $\sup A \geq 0$. (Why does $\sup A$ exist?)

(b) Prove that

$$\sup\left(\frac{1}{2}A\right) = \frac{1}{2}\sup A$$

Solution. (a) *Proof.* First, $\sup A$ exists by the Axiom of Completeness. For any $a \in A$ is an upper found for A so $\sup A \geq a \geq 0$. □

(b) Let $s = \sup A$. We show that $\frac{1}{2}s = \sup\left(\frac{1}{2}A\right)$. For any $x \in \frac{1}{2}A$, we have $2x \in A$, so $2x \leq s$ and $x \leq s/2$. So, $s/2$ is an upper bound for $\frac{1}{2}A$.

If b is an upper bound for $\frac{1}{2}A$, then $\frac{1}{2}a \leq b$ for all $a \in A$. But then $a \leq 2b$ for all $a \in A$, so $2b$ is an upper bound for A. Hence $s \leq 2b$ and $s/2 \leq b$.

This makes $s/2$ the least upper bound for $\frac{1}{2}A$.

Problem 4. (20 pts) Let (a_n) be a convergent sequence with $(a_n) \to a$ and let (b_n) be a convergent sequence with $(b_n) \to b$. By the Algebraic Limit Theorem, we know that $(a_n + b_n) \to a + b$.

Prove that $(a_n + b_n)$ converges to $a + b$ directly using the definition.

Solution. *Proof.* Let (a_n) be a convergent sequence with $(a_n) \to a$ and let (b_n) be a convergent sequence with $(b_n) \to b$.

Let $\epsilon > 0$. Since (a_n) converges to a, there exists N_1 so that $|a_n - a| < \epsilon/2$ whenever $n \geq N_1$. Similarly, since (b_n) converges to b, there exists N_2 so that $|b_n - b| < \epsilon/2$ whenever $n \geq N_2$. Let $N = \max\{N_1, N_2\}$ and let $n \geq N$. Then

$$\begin{aligned} |(a_n + b_n) - (a + b)| &= |(a_n - a) + (b_n - b)| \\ &\leq |a_n - a| + |b_n - b| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, $(a_n + b_n)$ converges to $a + b$ by definition. □

Problem 5. (20 pts) Let (a_n) be a monotone sequence and suppose that (a_n) has a convergent subsequence. Show that (a_n) converges.

Solution. *Proof.* Let (a_n) be a monotone sequence and suppose that (a_n) has a convergent subsequence (a_{n_k}) converging to ℓ. Without loss of generality, we can assume (a_n) is increasing.

Let $\epsilon > 0$. Since (a_{n_k}) converges to ℓ, there exists $K \in \mathbb{N}$ so that $|a_{n_k} - \ell| < \epsilon$ whenever $k \geq K$. Let $N = n_K$ and let $n \geq N$. Then

$$a_{n_K} \leq a_n \leq \ell.$$

But then $|a_n - \ell| < \epsilon$.

Since $\epsilon > 0$ is arbitrary, (a_n) converges to ℓ as well. □