Final Examination
MATH 3243
Wednesday, December 10, 2025

Directions: *This is the final examination for MATH 3243. The test contains six problems counting various point values each for a total of 250 points. The value of each problem is indicated with the problem. You must complete all the problems. You must show all your work clearly and completely in the spaces provided. You may use your book and your calculator, but you may not give assistance to or receive assistance from anyone. You may not use any online resources except the online text book. You may use computational tools such as MAPLE, Mathematica, Symbolab, Desmos, etc., to check your answers after you finish the test by hand. If you violate these rules, you will fail the course. Your test is due as a single PDF by 11:59 pm on Wednesday, December 10, in the Assignments Folder for the Final Examination on Course Den.*

Good luck.

My signature below indicates that I have read and understand the instructions printed above and I agree to abide by them.

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Name (printed):____________________________

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Question	Score	Out of
1(a)		15
(b)		15
(c)		15
2		35
3(a)		10
(b)		10
(c)		10
(d)		10
(e)		10
4(a)		20
(b)		20
5(a)		20
(b)		20
6(a)		20
(b)		20
Total		250

Problem 1. (45 pts) Let f and g denote functions defined on some set A.

a) Prove that

$$\sup_{x\in A}(f(x)+g(x)) \leq \sup_{x\in A} f(x) + \sup_{x\in A} g(x).$$

Solution. For any $x \in A$, $f(x) \leq \sup_{x\in A} f(x)$ and $g(x) \leq \sup_{x\in A} g(x)$ and hence $f(x)+g(x) \leq \sup_{x\in A} f(x)+\sup_{x\in A} g(x)$. Thus $\sup_{x\in A} f(x)+\sup_{x\in A} g(x)$ is an upper bound for $f(x)+g(x)$ on A, and hence it is no smaller than the least upper bound for $f(x)+g(x)$ on A, which is $\sup_{x\in A}(f(x)+g(x))$.

b) Find an example for a pair f, g for which

$$\sup_{x\in A}(f(x)+g(x)) = \sup_{x\in A} f(x) + \sup_{x\in A} g(x).$$

a) Take say f and g to be the constant functions 0, and then $\sup_{x\in A}(f(x)+g(x))$ and $\sup_{x\in A} f(x) + \sup_{x\in A} g(x)$ are both 0.

c) Find an example for a pair f, g for which

$$\sup_{x\in A}(f(x)+g(x)) < \sup_{x\in A} f(x) + \sup_{x\in A} g(x).$$

Solution. Take say $f(x) = x$ and $g(x) = -x$ on $A = [0,1]$. Then $f(x)+g(x) = 0$ and hence $\sup_{x\in A}(f(x)+g(x)) = 0$ while $\sup_{x\in A} f(x) = 1$ and $\sup_{x\in A} g(x) = 0$ and hence $\sup_{x\in A} f(x) + \sup_{x\in A} g(x) = 1$. Thus $\sup_{x\in A}(f(x)+g(x)) = 0 < 1 = \sup_{x\in A} f(x) + \sup_{x\in A} g(x)$ as required.

Problem 2. (35 pts) Sketch the graph of the function

$$y = f(x) = \frac{x^2}{x^2 - 1}.$$

Make sure that your graph clearly indicates the following:

- The domain of definition of $f(x)$.
- The behavior of $f(x)$ near the points where it is not defined (if any) and as $x \to \pm\infty$.
- The exact coordinates of the x- and y-intercepts and all minima and maxima of $f(x)$.

You can put your work on this page and the graph of the function on the next page.

Solution. The function $f(x)$ is defined for $x \neq \pm 1$, and the following limits are easily computed: $\lim_{x\to\pm\infty} f(x) = 1$, $\lim_{x\to -1^-} f(x) = \lim_{x\to -1^+} f(x) = 1$ and $\lim_{x\to -1^-} f(x) = \lim_{x\to -1^+} f(x) = 1$ The only solution for $f(x) = 0$ is $x = 0$, hence the only intersection of the graph of $f(x)$ with the axes is at $(0, 0)$. Other than at $x = 0$, the numerator of f is always positive, hence the sign of the function is determined by the sign of the denominator $x^2 - 14$. Thus $f(x) \leq 0$ for $|x| < 1$ and $f(x) \geq 0$ for $|x| > 1$. Finally

$$f'(x) = \frac{2x(x^2-1) - x^2 \cdot 2x}{(x^2-1)^2} = -\frac{2x}{(x^2-1)^2}.$$

and thus f' is positive and f is increasing (locally) for $x < 0$ and f' is negative and f is decreasing (locally) for $x > 0$.

Thus overall the graph is:

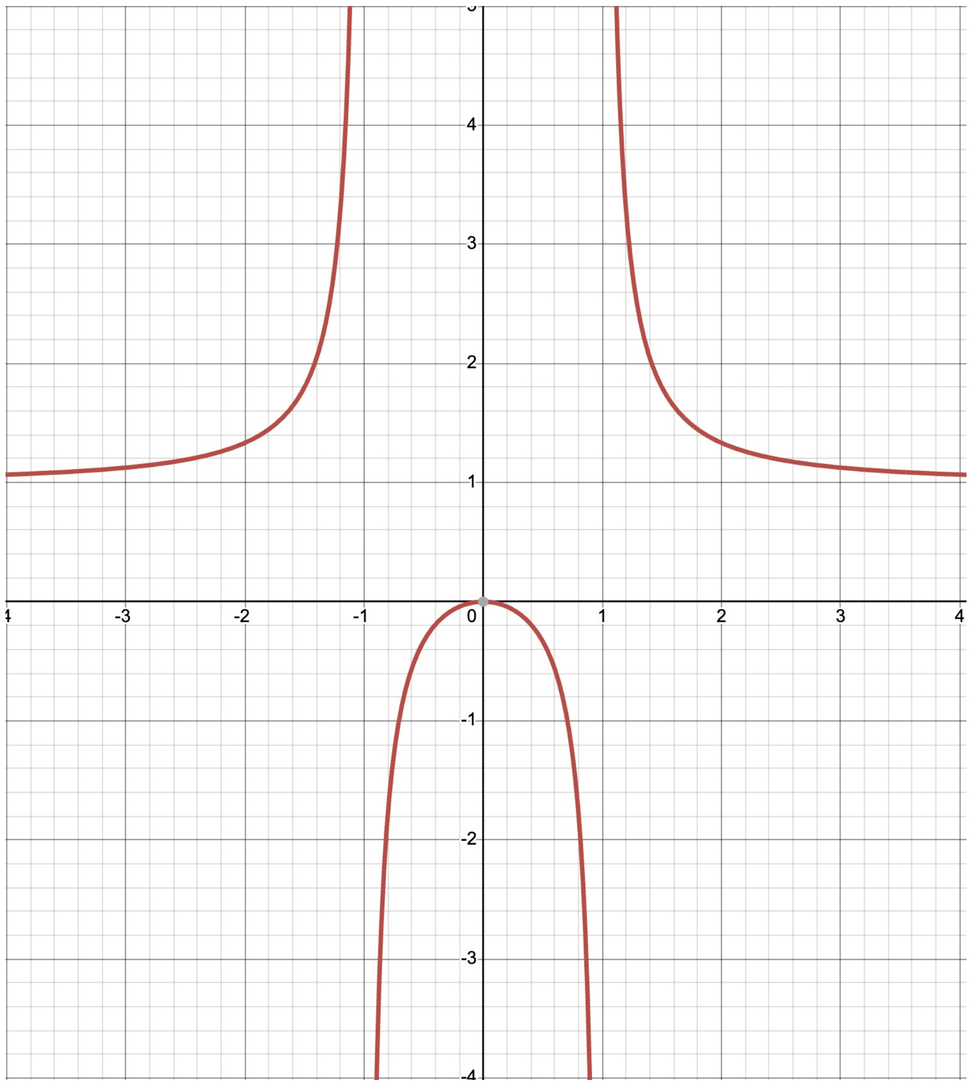

Figure 1: Sketch for Problem 2

Problem 3. (50 pts) Compute the following integrals:

a)

$$\int \frac{x^2+1}{x+1}\,dx$$

Solution. By long division of polynomials, $x^2+1=(x+1)(x-1)+2$. Thus we can rewrite our integral as a sum of two terms as follows

$$\begin{aligned}\int \frac{x^2+1}{x+1}\,dx &= \int \frac{(x+1)(x-1)+2}{x+1}\,dx \\ &= \int x-1+\frac{2}{x+1}\,dx \\ &= \int x-1\,dx + \int \frac{2}{x+1}\,dx \\ &= \frac{1}{2}x^2 - x + 2\ln|x+1| + C.\end{aligned}$$

b)

$$\int \frac{x+1}{x^2+1}\,dx$$

Solution. Again we rewrite the integral as a sum of two terms. On the first we perform the substitution $u = x^2$; the second is elementary:

$$\begin{aligned}
\int \frac{x+1}{x^2+1}\,dx &= \int \frac{x}{x^2+1} + \frac{1}{x^2+1}\,dx \\
&= \int \frac{x}{x^2+1}\,dx + \int \frac{1}{x^2+1}\,dx \\
&= \frac{1}{2}\int \frac{1}{u+1}\,du + \arctan x + C \\
&= \frac{1}{2}\ln|u+1| + \arctan x + C \\
&= \frac{1}{2}\ln(x^2+1) + \arctan x + C.
\end{aligned}$$

c)

$$\int x^2 \sin x \, dx$$

Solution. We integrate by parts letting $u = x^2$ and $dv = \sin x \, dx$. Then $du = 2x \, dx$ and $v = -\cos x$. Substituting, we get

$$\begin{aligned}\int x^2 \sin x \, dx &= -x^2 \cos x - 2 \int x(-\cos x) \, dx \\ &= -x^2 \cos x + 2 \int x \cos x \, dx\end{aligned}$$

We integrate by parts a second time letting $u = x$ and $dv = \cos x \, dx$. Then $du = dx$ and $v = \sin x$. Substituting, we get

$$\begin{aligned}\int x^2 \sin x \, dx &= -x^2 \cos x - 2 \left[x \sin x - \int \sin x \, dx \right] \\ &= -x^2 \cos x + 2x \sin x - 2 \int \sin x \, dx \\ &= -x^2 \cos x + 2x \sin x + 2 \cos x + C.\end{aligned}$$

d)

$$\int \frac{dx}{\sqrt{1+e^x}}$$

Solution. Set $u = \sqrt{1+e^x}$. Then $e^x = u^2 - 1$ and $du = \frac{e^x\,dx}{2\sqrt{1+e^x}} = \frac{(u^2-1)\,dx}{2u}$ and so $dx = \frac{2u\,du}{u^2-1}$. Substituting, we get

$$\begin{aligned}
\int \frac{dx}{\sqrt{1+e^x}} &= \int \frac{2u\,du}{u(u^2-1)} \\
&= \int \frac{2\,du}{u^2-1} \\
&= \int \frac{1}{u-1} - \frac{1}{u+1}\,du \\
&= \int \frac{1}{u-1}\,du - \int \frac{1}{u+1}\,du \\
&= \ln|u-1| - \ln|u+1| + C \\
&= \ln\left|\frac{u-1}{u+1}\right| + C \\
&= \ln\left|\frac{\sqrt{1+e^x}-1}{\sqrt{1+e^x}+1}\right| + C.
\end{aligned}$$

e)

$$\int_0^\infty e^{-x}\,dx$$

Solution. We compute

$$\begin{aligned}\int_0^\infty e^{-x}\,dx &= \lim_{b\to\infty}\int_0^b e^{-x}\,dx\\ &= \lim_{b\to\infty}\left[-e^{-x}\right|_0^b\\ &= \lim_{b\to\infty}(1-e^{-b})\\ &= 1.\end{aligned}$$

Problem 4. (40 pts) Agents of the NSA have secretly developed a function $e(x)$ that has the following properties:

- $e(x+y) = e(x)e(y)$ for all $x, y \in \mathbb{R}$.
- $e(0) = 1$.
- e is differentiable at 0 and $e'(0) = 1$.

Prove the following:

a) Prove that e is everywhere differentiable and $e' = e$.

Solution. The given fact that $1 = e'(0)$ means that

$$1 = \lim_{h \to 0} \frac{e(h) - e(0)}{h} = \lim_{h \to 0} \frac{e(h) - 1}{h}.$$

Hence, using $e(x+h) = e(x)e(h)$ we get

$$\lim_{h \to 0} \frac{e(x+h) - e(x)}{h} = \lim_{h \to 0} \frac{e(x)e(h) - e(x)}{h} = e(x) \lim_{h \to 0} \frac{e(h) - 1}{h} = e(x).$$

b) Prove that $e(x) = e^x$ for all $x \in \mathbb{R}$. The only lemma you may assume is that if a function f satisfies $f'(x) = 0$ for all x then f is a constant function.

Solution. Consider $q(x) = e(x)e^{-x}$. Differentiating we get

$$q'(x) = e'(x)e^{-x} + e(x)(e^{-x})' = e(x)e^{-x} - e(x)e^{-x} = 0.$$

Hence $q(x)$ is a constant function. But $q(0) = e(0)e^0 = 1 \cdot 1 = 1$, hence this constant must be 1. So $e(x)e^{-x} = 1$ and thus $e(x) = e^x$.

Problem 5. (40 pts)

a) Prove that if a sequence of continuous functions (f_n) converges uniformly to a function f on some interval $[a, b]$, then f is continuous on $[a, b]$.

Solution. See Theorem 6.2.6 (Continuous Limit Theorem) on pp. 178–179 of the text book.

b) Prove that the series $\sum_{n=1}^{\infty} \frac{1}{2^n} \sin(3^n x)$ converges on $(-\infty, \infty)$ and that its sum is a continuous function of x.

Solution. Since

$$\left| \frac{1}{2^n} \sin(3^n x) \right| \leq \frac{1}{2^n}$$

and $\sum_{n=1}^{\infty} \frac{1}{2^n}$ converges, by the Weierstrass M-Test the series $\sum_{n=1}^{\infty} \frac{1}{2^n} \sin(3^n x)$ converges uniformly. As each of the terms $\frac{1}{2^n} \sin(3^n x)$ is continuous, the first part of this question implies that so is the sum.

Problem 6. (40 pts) Let f be the complex function $f(z) = \overline{z}$.

a) Prove that f is everywhere continuous.

Solution. The key point is that $|w| = |\overline{w}|$ for every complex number w. Let $\epsilon > 0$ and set $\delta = \epsilon$. Now if $|z - z_0| < \delta$ then

$$|\overline{z} - \overline{z_0}| = |\overline{z - z_0}| = |z - z_0| < \delta = \epsilon.$$

This proves the continuity of $f(z) = \overline{z}$.

b) Prove that f is nowhere differentiable.

Solution. Let us check if this function is differentiable:

$$\lim_{h\to 0} \frac{\overline{z+h} - \overline{z}}{h} = \lim_{h\to 0} \frac{\overline{z} + \overline{h} - \overline{z}}{h} = \lim_{h\to 0} \frac{\overline{h}}{h}.$$

If we restrict our attention to real h then the latter quotient is always 1, so the limit would be 1. If we restrict our attention to imaginary h then the latter quotient is always -1, so the limit would be -1. Hence the limit cannot exist and $f(z) = \overline{z}$ is not differentiable at (an arbitrary) z.