Two Has a Real Square Root

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Fall Semester, 2025

Consider the set

$$\{t \in \mathbb{R} \mid t^2 < 2\}$$

The set t is bounded above. Let α be the least upper bound of T.

Think. If $\alpha^2 < 2$, we want a number $\alpha + 1/n$ so that

$$(\alpha + 1/n)^2 < 2.$$

This means $\alpha + 1/n \in T$ which is a contradiction.

So, we want

$$\alpha^2 + 2\alpha/n + (1/n)^2 < 2$$

We know that $(1/n)^2 < 1/n$, so if we make

$$\alpha^2 + 2\alpha/n + 1/n < 2,$$

we will have

$$\alpha^2 + 2\alpha/n + (1/n)^2 < \alpha^2 + 2\alpha/n + 1/n < 2$$

So, we want

$$\begin{aligned}
\alpha^2 + 2\alpha/n + 1/n &< 2 \\
(2\alpha + 1)/n &< 2 - \alpha^2 \\
1/n &< (2 - \alpha^2)/(2\alpha + 1).
\end{aligned}$$

So, choose

$$\frac{1}{n} < \frac{2 - \alpha^2}{2\alpha + 1}.$$

Think. If $\alpha^2 > 2$, we want a number $\alpha - 1/n$ so that

$$(\alpha - 1/n)^2 > 2.$$

This means there is no number in T between $\alpha - 1/n$ and α, which is a contradiction.

So, we want

$$\alpha^2 - 2\alpha/n + (1/n)^2 > 2$$

We know that $\alpha^2 - 2\alpha/n + (1/n)^2 > \alpha^2 - 2\alpha/n$, so if we make

$$\alpha^2 - 2\alpha/n > 2,$$

we will have

$$\alpha^2 - 2\alpha/n + (1/n)^2 < \alpha^2 - 2\alpha/n > 2$$

So, we want

$$\begin{aligned} \alpha^2 - 2\alpha/n &> 2 \\ \alpha^2 - 2 &> 2\alpha/n \\ 1/n &< (\alpha^2 - 2)/(2\alpha). \end{aligned}$$

So, choose

$$\frac{1}{n} < \frac{\alpha^2 - 2}{2\alpha}.$$

Proof. Suppose $\alpha^2 > 2$. Choose $n \in \mathbb{N}$ so that

$$\frac{1}{n} < \frac{\alpha^2 - 2}{2\alpha}.$$

Then

$$\begin{aligned} (\alpha - 1/n)^2 &= \alpha^2 - \frac{2\alpha}{n} + \frac{1}{n^2} \\ &> \alpha^2 - \frac{2\alpha}{n} \\ &> \alpha^2 - (\alpha^2 - 2) = 2. \end{aligned}$$

Hence, $\alpha - 1/n \notin T$, so there is no number β with $\alpha - 1/n < \beta \leq \alpha$ in T. This says $\alpha - 1/n$ is an upper bound for T, a contradiction. $\square$