

Rationals Between Reals

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Theorem. *Let $a, b \in \mathbb{R}$ with $a < b$. Then there exists a rational number r so that $a < r < b$.*

Think. We want $r = m/n$ with

$$a < m/n < b.$$

So, we want

$$an < m < bn$$

So, let m be the smallest natural number greater than an . Then

$$\begin{aligned} m - 1 &\leq an \\ m &\leq an + 1 \end{aligned}$$

and we want this less than bn . So, we want

$$\begin{aligned} an + 1 &< bn \\ 1 &< n(b - a) \\ 1/n &< b - a. \end{aligned}$$

So, let's start by choosing $n \in \mathbb{N}$ so that $1/n < b - a$, which we can do by the Archimedean property since $b - a > 0$.