Differentiable Limit Theorem

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Theorem (Differentiable Limit Theorem)

Let $f_n \to f$ pointwise on the closed interval $[a, b]$, and assume that each f_n is differentiable. If (f_n') converges uniformly on $[a, b]$ to a function g, then the function f is differentiable and $f' = g$.

This says convergence preserves differentiability of functions provided the sequence of functions converges pointwise and the sequence of derivatives converges uniformly.

Proof

Proof

Let $f_n \to f$ pointwise on the closed interval $[a, b]$, and suppose that each f_n is differentiable. Further suppose that If (f_n') converges uniformly on $[a, b]$ to a function g.

Notice we state the hypotheses first.

Proof (cont.)

Proof

Let $c \in [a, b]$ be fixed. We want $f'(c)$ to equal $g(c)$.
Let $\epsilon > 0$.

We fix c from the start because differentiability happens one point at a time.

Almost every proof involving an epsilon-delta argument begins with the sentence "Let $\epsilon > 0$."

Proof (cont.)

Proof

First we write

$$\left|\frac{f(x)-f(c)}{x-c}-g(c)\right| \leq \left|\frac{f(x)-f(c)}{x-c}-\frac{f_n(x)-f_n(c)}{x-c}\right| + \left|\frac{f_n(x)-f_n(c)}{x-c}-f_n'(c)\right| + \left|f_n'(c)-g(c)\right|$$

This is giving insight into how the proof will proceed. We will make each of these last three quantities less than $\epsilon/3$.

Proof (cont.)

Proof

First we write

$$\left|\frac{f(x)-f(c)}{x-c}-g(c)\right| \leq \left|\frac{f(x)-f(c)}{x-c}-\frac{f_n(x)-f_n(c)}{x-c}\right| + \left|\frac{f_n(x)-f_n(c)}{x-c}-f_n'(c)\right| + \left|f_n'(c)-g(c)\right|$$

We use the pointwise convergence of (f_n) and the uniform convergence of (f_n') to find an f_n that forces the first and third terms to be less than $\epsilon/3$.

Proof (cont.)

Proof

First we write

$$\left|\frac{f(x)-f(c)}{x-c}-g(c)\right| \leq \left|\frac{f(x)-f(c)}{x-c}-\frac{f_n(x)-f_n(c)}{x-c}\right| + \left|\frac{f_n(x)-f_n(c)}{x-c}-f_n'(c)\right| + \left|f_n'(c)-g(c)\right|$$

Once we find that f_n, we can then use differentiability of f_n to produced a δ that makes the middle term less than $\epsilon/3$ for all x satisfying $0 < |x-c| < \delta$.

Proof (cont.)

Proof

First choosing $N_1 \in \mathbb{N}$ so that

$$\left| f_m'(c) - g(c) \right| < \frac{\epsilon}{3} \tag{1}$$

for all $m \geq N_1$.

We can do this because $(f_n'(c))$ converges $g(c)$.

Proof (cont.)

Proof

Since (f_n') converges uniformly, the Cauchy Criterion for Uniform Convergence says we can find N_2 so that whenever n, $m \geq N_2$,

$$|f_m'(x) - f_n'(x)| < \frac{\epsilon}{3}$$

for all $x \in [a, b]$.

We can do this because (f_n') converges uniformly on $[a, b]$. This statement is saying the sequence (f_n') is uniformly Cauchy on $[a, b]$.

Proof (cont.)

Proof

Let $N = \max\{N_1, N_2\}$.

We want both of the previous conditions to happen, so we want N at least as big as each of them.

Proof (cont.)

Proof

Since f_N is differentiable, there exists $\delta > 0$ so that

$$\left| \frac{f_N(x) - f_N(c)}{x - c} - f_N'(c) \right| < \frac{\epsilon}{3} \tag{2}$$

whenever $0 < |x - c| < \delta$.
This is the δ we want, but it takes some effort to show that.

Now that we have one f_N fixed, we can use the fact that f_N is differentiable at c to find this δ. Now we have to show this δ does all the things we want it to do.

Proof (cont.)

Proof

Fix x satisfying $0 < |x - c| < \delta$.

Let $h(x) = f_m(x) - f_N(x)$.

Applying the Generalized Mean Value Theorem to h, we get

$$\frac{h(x) - h(c)}{x - c} = h'(\alpha)$$

for some α between c and x.

Notice h satisfies the hypotheses of the Generalized Mean Value Theorem since the sequence (f_n) does.

Proof (cont.)

Proof

That is,

$$\begin{aligned}&\frac{f_m(x) - f_N(x) - (f_m(c) - f_N(c))}{x - c} \\ &= \frac{f_m(x) - f_m(c) - (f_N(x) - f_N(c))}{x - c} \\ &= f'_m(\alpha) - f'_N(\alpha).\end{aligned}$$

This just substitutes the definition of h into the last result. Notice

$$\tfrac{f_m(x) - f_N(x) - (f_m(c) - f_N(c))}{x - c} = \tfrac{f_m(x) - f_m(c) - (f_N(x) - f_N(c))}{x - c}.$$

Proof (cont.)

Proof

So

$$\left|\frac{f_m(x)-f_m(c)}{x-c}-\frac{f_N(x)-f_N(c)}{x-c}\right|<\frac{\epsilon}{3}.$$

Letting $m\to\infty$, the Order Limit Theorem gives us

$$\left|\frac{f(x)-f(c)}{x-c}-\frac{f_N(x)-f_N(c)}{x-c}\right|\leq\frac{\epsilon}{3}. \qquad (3)$$

The first inequality comes from the choice of N provided $m \geq N$. (See slide 11 where we chose N_2.)

The second inequality is just an application of the Order Limit Theorem to the first inequality.

Proof (cont.)

Proof

The inequalities (1), (2), and (3), together imply that for x satisfying $0 < |x - c| < \delta$

$$\left|\frac{f(x) - f(c)}{x - c} - g(c)\right| \leq \left|\frac{f(x) - f(c)}{x - c} - \frac{f_N(x) - f_N(c)}{x - c}\right| + \left|\frac{f_N(x) - f_N(c)}{x - c} - f_N'(c)\right| + \left|f_N'(c) - g(c)\right| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.$$

We have found each of these expressions is less than $\epsilon/3$. So, put it all together.

Proof (cont.)

Proof

Since x satisfying $0 < |x - c| < \delta$ is arbitrary, we have shown that

$$\left| \frac{f(x) - f(c)}{x - c} - g(c) \right| < \epsilon$$

for all $0 < |x - c| < \delta$.
So,

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = g(c).$$

which says $f'(c) = g(c)$.

This is just the definition of limit.

Proof (cont.)

Proof

Of course, since c is arbitrary, this proves the result for all c:

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = g(c).$$

which says $f'(c) = g(c)$ for all c.

We fixed c initially so our computation would prove the result for all c.