Continuous Limit Theorem

Understanding Analysis by Stephen Abbott

slideshow by
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Goal of the Lecture

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We will initially investigate the convergence of sequences of functions in preparation for dealing with convergence of series of functions, particularly power series.

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In this lecture, we will examine pointwise convergence of a sequence of functions, discover that pointwise convergence doesn't preserve nice properties of sequences of functions. Then we will define and develop a stronger form of convergence of a sequence of functions: uniform convergence.

In particular, we will determine what conditions are needed for sequences to preserve continuity.

Pointwise Convergence

Pointwise Convergence

Definition

For each $n \in \mathbb{N}$, let f_n be a function defined on a set $A \subseteq \mathbb{R}$. The sequence (f_n) of function *converges pointwise on* A to a function $f : A \to \mathbb{R}$ if, for all $x \in A$, the sequence of real numbers $(f_n(x))$ converges to $f(x)$.

In the definition of pointwise convergence, the rate at which the sequence $(f_n(x))$ converges to $f(x)$ depends on x.

Pointwise Convergence

The problem with pointwise convergence is that certain nice properties of functions are not preserved under pointwise convergence.

For example, if a sequence of continuous functions converges pointwise, the limit may not be continuous.

And if a sequence of differentiable functions converges pointwise, the limit may not be differentiable.

Non-Example 1

Pointwise Convergence

Example

Let

$$g_n(x) = x^n$$

on the set $[0, 1]$. Notice that each g_n is a polynomial so it is continuous on $[0, 1]$.

For $0 \leq x < 1$, $g_n(x)$ converges to 0.

For $x = 1$, $x^n = 1$, so $g_n(x)$ converges to 1.

Pointwise Convergence

Example

So, (g_n) converges to the function

$$g = \begin{cases} 0 & \text{for } 0 \leq x < 1, \\ 1 & \text{for } x = 1. \end{cases}$$

So, we have each g_n is continuous on $[0, 1]$, but its limit g is not continuous at $x = 0$.

Non-Example 2

Pointwise Convergence

Example

Let

$$h_n(x) = x^{1+\frac{1}{2n-1}}$$

on the set $[-1, 1]$. Notice that each h_n is differentiable on $[-1, 1]$.

Pointwise Convergence

Example

Note that
$$h_n(x) = x^{1+\frac{1}{2n-1}} = x \cdot x^{\frac{1}{2n-1}}.$$

For x in the inteval $[-1, 1]$, when you take a larger and larger odd root of x, that root goes to 1 if $x > 0$ and -1 if $x < 0$.

So, for $x \geq 0$, $h_n(x)$ converges to x and for $x < 0$, $h_n(x)$ converges to $-x$.

Pointwise Convergence

Example

So, (h_n) converges to the function

$$h(x) = |x|$$

So, we have each h_n is differentiable on $[-1, 1]$, but its limit h is not differentiable at $x = 0$.

The problem with this definition of functional convergence is that value of n depends on x.

Clearly we need a stronger version of convergence than pointwise convergence to make these nice properties carry through to the limits of the sequences.

We need one value of n to work for all x simultaneously.

Uniform Convergence

Uniform Convergence

Definition

Let (f_n) be a sequence of functions defined on a set $A \subseteq \mathbb{R}$. Then, (f_n) *converges uniformly on* A to a limit function f defined on A if, for every $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that $|f_n(x) - f(x)| < \epsilon$ whenever $n \geq N$ and $x \in A$.

In the definition of uniform convergence, the rate at which the sequence (f_n) converges to f depends only on n, not on x.

Cauchy Criterion for Uniform Convergence

Cauchy Criterion for Uniform Convergence

Theorem (Cauchy Criterion for Uniform Convergence)

A sequence of functions (f_n) defined on a set $A \subseteq \mathbb{R}$ converges uniformly on A if and only if for every $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that $|f_n(x) - f_m(x)| < \epsilon$ for all $m, n \geq N$ and all $x \in A$.

This says each sequence $(f_n(x))$ is a Cauchy sequence and the same N works for all $x \in A$. That is, N only depends on the sequence of functions, not on any specific point of A.

Such a sequence is sometimes described as "uniformly Cauchy."

Continuous Limit Theorem

Continuous Limit Theorem

Theorem (Continuous Limit Theorem)

Let (f_n) be a sequence of functions defined on $A \subseteq \mathbb{R}$ that converges uniformly on A to a function f. If each f_n is continuous at $c \in A$, then f continuous at c.

This says uniform convergence preserves continuity of functions.

Continuous Limit Theorem

Proof

Let (f_n) be a sequence of functions on A which converges uniformly to a function f on A. Further assume, each f_n is continuous.

Notice we state the hypotheses first.

Continuous Limit Theorem

Proof

Fix $c \in A$ and let $\epsilon > 0$.

We fix c from the start because continuity happens one point at a time.

Almost every proof involving an epsilon-delta argument begins with the sentence "Let $\epsilon > 0$."

Continuous Limit Theorem

Proof

Choose N so that

$$|f_N(x) - f(x)| < \frac{\epsilon}{3}$$

for all $x \in A$.

Here we have used the fact that the sequence (f_n) converges to f uniformly on A. This allows us to choose one f_N which is uniformly within $\epsilon/3$ of f.

Continuous Limit Theorem

Proof

Because f_N is continuous, there exists a $\delta > 0$ for which

$$|f_N(x) - f_N(c)| < \frac{\epsilon}{3}$$

is true whenever $|x - c| < \delta$.
Fix $|x - c| < \delta$.

We use the continuity of f_N to find a δ-neighborhood of c so that the distance from $f_N(x)$ to $f_N(c)$ is small. We then fix x in this δ-neighborhood of c.

Continuous Limit Theorem

Proof

This implies

$$
\begin{aligned}
|f(x) - f(c)| &= |f(x) - f_N(x) + f_N(x) - f_N(c) + f_N(c) - f(c)| \\
&\leq |f(x) - f_N(x)| + |f_N(x) - f_N(c)| + |f_N(c) - f(c)| \\
&< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\
&< \epsilon.
\end{aligned}
$$

The first and third inequalities come from the uniform convergence of (f_n) by the choice of N. The middle inequality comes from the continuity of f_N and the fact that x lies in the δ-neighborhood of c.

Continuous Limit Theorem

Proof

Since $\epsilon > 0$ is arbitrary, f is continuous at c.
Since $c \in A$ is arbitrary, f is is continuous on A. $\square$

This concludes the proof showing that f is is continuous on A.