Homework #9 Solutions
Due Monday, October 13

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Exercise 4.3.3. (a) Supply a proof for Theorem 4.3.9 using the ϵ-δ characterization of continuity.

(b) Give another proof of this theorem using the sequential characterization of continuity (from Theorem 4.3.2 (iii)).

Solution.

Theorem (Composition of Continuous Functions)**.** *Given $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$, assume that the range $f(A) = \{f(x) : x \in A\}$ is contained in the domain B so that the composition $g \circ f(x) = g(f(x))$ is well-defined on A.*

If f is continuous at $c \in A$, and if g is continuous at $f(c) \in B$, then $g \circ f$ is continuous at c.

(a) *Proof.* Let $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$ be continuous on their domains, assume that the range $f(A) = \{f(x) : x \in A\}$ is contained in the domain B. Suppose f is continuous at $c \in A$, and if g is continuous at $f(c) \in B$.

Let $\epsilon > 0$. Since g is continuous at $f(c)$, there exists $\eta > 0$ so that $|g(b) - g(f(c))| < \epsilon$ provided $|b - f(c)| < \eta$. Since f is continuous at c, there exists $\delta > 0$ so that $|f(x) - f(c))| < \eta$ provided $|x - c| < \delta$.

Let $|x - c| < \delta$. Then $|f(x) - f(c))| < \eta$, whereby $|g(f(x)) - g(f(c))| < \epsilon$. Since $\epsilon > 0$ is arbitrary, $g \circ f$ is continuous at c. □

(b) *Proof.* Let $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$ be continuous on their domains, assume that the range $f(A) = \{f(x) : x \in A\}$ is contained in the domain B. Suppose f is continuous at $c \in A$, and if g is continuous at $f(c) \in B$.

Let (x_n) be an sequence in A converging to c. Since f is continuous at c, by the sequential characterization of continuity (Theorem 4.3.2), $(f(x_n))$ converges to $f(c)$. Since g is continuous at $f(c)$ and $(f(x_n))$ converges to $f(c)$, by the sequential characterization of continuity (Theorem 4.3.2), $(g(f(x_n)))$ converges to $g(f(c))$. Since the sequence (x_n) in A converging to c is arbitrary, $g \circ f$ is continuous at c, again by sequential characterization of continuity (Theorem 4.3.2). □

Exercise 4.3.9. Assume $h : \mathbb{R} \to \mathbb{R}$ is continuous on $\mathbb{R}$ and let $K = \{x : h(x) = 0\}$. Show that K is a closed set.

Solution. *Proof.* Let $h : \mathbb{R} \to \mathbb{R}$ be continuous on $\mathbb{R}$ and let $K = \{x : h(x) = 0\}$.

The set $\{0\} \subset \mathbb{R}$ is a closed set. Since h is continuous, $h^{-1}(\{0\})$ is also a closed set. But this is exactly the set K. So, K is a closed set. $\square$

Exercise 4.3.11 (Contraction Mapping Theorem)**.** Let f be a function defined on all of $\mathbb{R}$, and assume there is a constant c such that $0 < c < 1$ and

$$|f(x) - f(y) \leq c|x - y|$$

for all x, $y \in \mathbb{R}$.

(a) Show that f is continuous on $\mathbb{R}$.

(b) Pick some point $y_1 \in \mathbb{R}$ and construct the sequence

$$(y_1, f(y_1), f(f(y_1)), \dots).$$

In general, if $y_{n+1} = f(y_n)$, show that the resulting sequence (y_n) is a Cauchy sequence. Hence we may let $y = \lim y_n$.

(c) Prove that y is a fixed point of f (i.e., $f(y) = y$) and that it is unique in this regard.

(d) Finally, prove that if x is *any* arbitrary point in $\mathbb{R}$, then the sequence $(x, f(x), f(f(x)), \dots)$ converges to y defined in (b).

Solution. Let f be a function defined on all of $\mathbb{R}$, and assume there is a constant c such that $0 < c < 1$ and

$$|f(x) - f(y)| \leq c|x - y|$$

for all x, $y \in \mathbb{R}$.

(a) Let $x \in \mathbb{R}$ be arbitrary. Let $\epsilon > 0$. Let $0 < \delta < \epsilon/c$ and let $|x - y| < \delta$. Then

$$|f(x) - f(y)| \leq c|x - y| < c \cdot \delta < \epsilon.$$

Since $\epsilon > 0$ is arbitrary, f is continuous at x, and since $x \in \mathbb{R}$ is arbitrary, f is continuous on $\mathbb{R}$. (Since δ depends only on ϵ and not x, this actually shows f is uniformly continuous on $\mathbb{R}$.)

(b) Pick some point $y_1 \in \mathbb{R}$ and construct the sequence

$$(y_1, f(y_1), f(f(y_1)), \dots).$$

In general, if $y_{n+1} = f(y_n)$.

We first prove that

$$|y_n - y_{n-1}| \leq c^{n-2}|y_2 - y_1|$$

for all $n \geq 2$, the case $n = 2$ being trivial.

Assume

$$|y_n - y_{n-1}| \leq c^{n-2}|y_2 - y_1|$$

for some $n \in \mathbb{N}$ with $n \geq 2$. Then

$$\begin{aligned}|y_{n+1} - y_n| &= |f(y_n) - f(y_{n-1})| \\ &\leq c|y_n - y_{n-1}| \\ &\leq c \cdot c^{n-2}|y_2 - y_1| \\ &\leq c^{n-1}|y_2 - y_1|.\end{aligned}$$

By the Principle of Mathematical Induction, the result holds for all natural numbers $n \geq 2$.

For natural numbers $n > m$, we have

$$\begin{aligned}|y_n - y_m| &= \left|\sum_{k=m+1}^{n} y_k - y_{k-1}\right| \\ &\leq \sum_{k=m+1}^{n} |y_k - y_{k-1}| \\ &\leq \sum_{k=m+1}^{n} c^{k-2}|y_2 - y_1| \\ &\leq |y_2 - y_1| \sum_{k=m+1}^{n} c^{k-2}. \end{aligned} \tag{1}$$

Let $\epsilon > 0$. Since $0 < c < 1$, the sequence $\sum_{k=0}^{\infty} c^k$ is a convergent geometric series, so by the Cauchy Criterion for Series, there exists $N \in \mathbb{N}$ so that if $n > m \geq N$ then

$$c^{m+1} + \cdots + c^n < \frac{\epsilon}{|y_2 - y_1|}. \tag{2}$$

Finally, let $n > m \geq N + 2$. From Equations (1) and (2), we have

$$|y_n - y_m| \leq |y_2 - y_1| \sum_{k=m+1}^{n} c^{k-2} < |y_2 - y_1| \cdot \frac{\epsilon}{|y_2 - y_1|} = \epsilon.$$

This proves the sequence (y_n) is a Cauchy sequence.

Since this sequence is a Cauchy sequence, it converges by the Cauchy Criterion (Theorem 2.6.4). Let $y = \lim y_n$.

(c) We prove first that y is a fixed point of f. Since f is continuous, we have

$$f(y) = f(\lim_{n\to\infty} y_n) = \lim_{n\to\infty} f(y_n) = \lim_{n\to\infty} y_{n+1} = y.$$

We next prove that the point y is the unique fixed point. Suppose y' is also a fixed point of f. Then

$$|y - y'| = |f(y) - f(y')| \leq c|y - y'|,$$

and since $0 < c < 1$, this is a contradiction unless $y = y'$. So the fixed point is unique.

(d) Let $x \in \mathbb{R}$ be arbitrary. By part (b), the sequence

$$(x, f(x), f(f(x)), \dots)$$

converges to some limit $\ell \in \mathbb{R}$. By part (c), ℓ is the unique fixed point of f. Since y is also a fixed point of f, we have $\ell = y$. This proves the result.

Exercise 4.4.3. Show that $f(x) = 1/x^2$ is uniformly continuous on the set $[1, \infty)$ but not on the set $(0, 1]$.

Solution. *Proof.* Let $f(x) = 1/x^2$. Let $\epsilon > 0$. We wish to choose $\delta > 0$ so that

$$|f(x) - f(y)| = \left| \frac{1}{x^2} - \frac{1}{y^2} \right| < \epsilon$$

whenever $|x - y| < \delta$, where δ depends only on ϵ.

Suppose we've chosen δ and let $|x - y| < \delta$. Then

$$\begin{aligned} |f(x) - f(y)| &= \left| \frac{1}{x^2} - \frac{1}{y^2} \right| \\ &= \left| \frac{y^2 - x^2}{x^2 y^2} \right| \\ &= \left| \frac{(y - x)(y + x)}{x^2 y^2} \right| \\ &= \frac{|y - x|\,|y + x|}{x^2 y^2} \\ &\le \delta \frac{|y + x|}{x^2 y^2}. \end{aligned}$$

We now need an estimate on how large $|x + y|/x^2 y^2$ is for x, $y \ge 1$. For x, $y \ge 1$, we have

$$\frac{x + y}{x^2 y^2} = \frac{1}{x y^2} + \frac{1}{x^2 y} \le 1 + 1 = 2.$$

Now we can finish the proof.

Let $\delta = \epsilon/2$ and let $|x - y| < \delta$. Then

$$\begin{aligned} |f(x) - f(y)| &= \left| \frac{1}{x^2} - \frac{1}{y^2} \right| \\ &= \left| \frac{y^2 - x^2}{x^2 y^2} \right| \\ &= \left| \frac{(y - x)(y + x)}{x^2 y^2} \right| \\ &= \frac{|y - x|\,|y + x|}{x^2 y^2} \\ &< \delta \frac{|y + x|}{x^2 y^2} \\ &< \delta \cdot 2 = \epsilon. \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, f is uniformly continuous on $[1, \infty)$.

To show that f is not uniformly continuous on $(0, 1]$, we use Theorem 4.4.6 (Sequential Criterion for Nonuniform Continuity). Let $x_n = 1/n$ and $y_n = 1/n^2$. We note that x_n, $y_n \in (0, 1]$ for all $n \in \mathbb{N}$. We compute

$$|x_n - y_n| = \frac{1}{n} - \frac{1}{n^2} = \frac{n-1}{n^2},$$

which goes to zero at n goes to infinity. However,

$$|f(x_n) - f(y_n)| = |n^2 - n^4| = n^4 - n^2 = n^2(n^2 - 1) \geq 12,$$

for all $n \geq 2$. By Theorem 4.4.5 (Sequential Criterion for Absence of Uniform Continuity), f is not uniformly continuous on $(0, 1]$. □

Exercise 4.4.4. Decide whether each of the following statements is true or false, justifying each conclusion.

(a) If f is continuous on $[a, b]$ with $f(x) > 0$ for all $a \leq x \leq b$, then $1/f$ is bounded on $[a, b]$ (meaning $1/f$ has bounded range).

(b) If f is uniformly continuous on a bounded set A, then $f(A)$ is bounded.

(c) If f is defined on $\mathbb{R}$ and $f(K)$ is compact whenever K is compact, then f is continuous on $\mathbb{R}$.

Solution. (a) This statement is true.

Since f is continuous on the compact set $[a, b]$, this function has a minimum m and a maximum M on this set. Since $f(x) > 0$ for all $a \leq x \leq b$, we have $0 < m \leq f(x) \leq M$ for all $a \leq x \leq b$. But then $1/M \leq 1/f(x) \leq 1/m$ for all $a \leq x \leq b$, so $1/f$ is bounded.

(b) This statement is true.

Since f is uniformly continuous on a bounded set A, f can be extended to a continuous function $\overline{f}$ on its closure $\overline{A}$. (You should prove this.) Since A is bounded, $\overline{A}$ is both closed and bounded, hence compact. Since $\overline{f}$ is continuous and $\overline{A}$ is compact, the set $\overline{f}(\overline{A})$ is compact, hence closed and bounded. Since $f(A) \subset \overline{f}(\overline{A})$ it, too, is bounded.

(c) This statement is false.

Define

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

for all $x \in \mathbb{R}$. The image of *any* set in $\mathbb{R}$ is a subset of $\{0, 1\}$, which is a compact set. However, f is not continuous anywhere.

Exercise 4.4.9 (Lipschitz Functions). A function $f : A \to \mathbb{R}$ is called *Lipschitz* if there exists a bound $M > 0$ such that

$$\left|\frac{f(x) - f(y)}{x - y}\right| \leq M$$

for all x, $y \in A$. Geometrically speaking, a function f is Lipschitz if there is a uniform bound on the magnitude of the slopes of lines drawn through any two points on the graph of f.

(a) Show that if $f : A \to \mathbb{R}$ is Lipschitz, then it is uniformly continuous on A.

(b) Is the converse statement true? Are all uniformly continuous functions necessarily Lipschitz?

Solution. (a) *Proof.* Let $f : A \to \mathbb{R}$ be Lipschitz. Since f is Lipschitz, there exists a Lipschitz constant M so that

$$\left|\frac{f(x) - f(y)}{x - y}\right| \leq M.$$

Without loss of generality, we assume $M > 0$. (Otherwise this problem is really boring.)

Let $\epsilon > 0$. Choose $\delta = \epsilon/M$. Suppose $|x - y| < \delta$. Then

$$|f(x) - f(y)| \leq M\,|x - y| < M \cdot \delta = \epsilon.$$

Since $\epsilon > 0$ is arbitrary, f is uniformly continuous. □

(b) No, not every uniformly continuous function is Lipschitz.

Let $f(x) = \sqrt{x}$ for $x \in A = (0, 1]$. The function f is continuous on the compact set $[0, 1]$, so it's uniformly continuous there. So, it's certainly uniformly continuous on the smaller set $A = (0, 1]$.

On the other hand, f is not Lipschitz.

For any Lipschitz function, the derivative is bounded. Let's see this. Suppose g is Lipschitz with Lipschitz constant M. Then

$$|g'(x)| = \lim_{y \to x}\left|\frac{g(y) - g(x)}{y - x}\right| \leq M.$$

Computing f', we have

$$f'(x) = \frac{1}{2\sqrt{x}}.$$

We see that f' is not bounded on the interval $(0, 1]$, so f is not Lipschitz on this interval.