# Homework #8 Solutions
# Due Monday, October 6

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**Exercise 3.3.4.** Assume $K$ is compact and $F$ is closed. Decide if the following sets are definitely compact, definitely closed, both, or neither.

(a) $K \cap F$

(b) $\overline{F^c \cup K^c}$

(c) $K \setminus F = \{x \in K : x \notin F\}$

(d) $\overline{K \cap F^c}$

**Solution.** (a) The set $K \cap F$ is a closed subset of a compact set, and is therefore both closed and compact.

(b) The set $\overline{F^c \cup K^c}$ is the closure of a set, so it is closed. Since $K$ is compact, $K$ is bounded, so $K^c$ is not bounded. Hence, $\overline{F^c \cup K^c}$ is not bounded and therefore not compact.

(c) If $K = [0, 1]$ and $F = [1/3, 2/3]$, then

$$K \setminus F = [0, 1/3) \cup (2/3, 1],$$

which is neither closed nor compact.

(d) Since $K \cap F^c \subseteq K$ and $K$ is closed, so $\overline{K \cap F^c} \subseteq K$ is the closed subset of a compact set. It is therefore closed and compact.

**Exercise 3.3.5.** Decide whether the following propositions are true or false. If the claim is valid, supply a short proof, and if the claim is false, provide a counterexample.

(a) The arbitrary intersection of compact sets is compact.

(b) The arbitrary union of compact sets is compact.

(c) Let $A$ be arbitrary, and let $K$ be compact. Then, the intersection $A \cap K$ is compact.

(d) If $F_1 \supseteq F_2 \supseteq F_3 \supseteq F_4 \supseteq \cdots$ is a nested sequence of nonempty closed sets, then the intersection $\cap_{n=1}^{\infty} F_n \neq \emptyset$.

**Solution.** (a) This statement is true. Every compact set is closed and the arbitrary intersection of closed sets is also closed. Every compact set is bounded and the intersection is contained in all the compact sets, so the intersection is also bounded. Since the intersection of compact sets is both closed and bounded, it's compact by the Heine-Borel Theorem.

(b) This is false. Let $A_n = [-n, n]$ for each $n \in \mathbb{N}$. Each $A_n$ is closed and bounded, so it is compact by the Heine-Borel Theorem. However $\bigcup_{n=1}^{\infty} A_n = \mathbb{R}$, which is not compact.

(c) This is false. Let $A = (0, 1)$ and let $K = [0, 1]$. Then $A \subseteq \mathbb{R}$, $K$ is compact (since it's closed and bounded), but the intersection is $(0, 1)$, which is not closed, so it cannot be compact.

(d) This is false. Let $F_n = [n, \infty)$, which is closed in $\mathbb{R}$. Then $F_1 \supseteq F_2 \supseteq F_3 \supseteq F_4 \supseteq \cdots$, but the intersection $\bigcap_{n=1}^{\infty} F_n$ is empty.

**Exercise 4.2.5.** Use Definition 4.2.1 to supply a proof for the following limit statements.

(a) $\lim_{x\to 2}(3x+4) = 10$.

(b) $\lim_{x\to 0} x^3 = 0$.

(c) $\lim_{x\to 2}(x^2 + x - 1) = 5$.

(d) $\displaystyle\lim_{x\to 3} \frac{1}{x} = \frac{1}{3}$,

**Solution.** (a) *Proof.* Let $\epsilon > 0$. Choose $\delta = \epsilon/3$. For $x$ with $0 < |x-2| < \delta$, we have

$$\begin{aligned} |f(x) - L| &= |(3x+4) - 10| \\ &= |3x - 6| \\ &= 3|x-2| \\ &< 3\delta \\ &< \epsilon. \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, $\lim_{x\to 2}(3x+4) = 10$. □

(b) *Proof.* Let $\epsilon > 0$. Choose $\delta = \sqrt[3]{\epsilon}$. For $x$ with $0 < |x| < \delta$, we have

$$\begin{aligned} |f(x) - L| &= |x^3 - 0| \\ &= |x^3| \\ &= |x|^3 \\ &< \delta^3 \\ &< \epsilon. \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, $\lim_{x\to 0} x^3 = 0$. □

(c) *Proof.* Let $\epsilon > 0$. Choose $\delta = \min\{\epsilon/6, 1\}$. For $x$ with $0 < |x-2| < \delta$, we have

$$\begin{aligned} |f(x) - L| &= |(x^2 + x - 1) - 5| \\ &= |x^2 + x - 6| \\ &= |(x+3)(x-2)| \\ &< 6\delta \\ &< \epsilon. \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, $\lim_{x\to 2}(x^2 + x - 1) = 5$.

□

(d) *Proof.* Let $\epsilon > 0$. Choose $\delta = \min\{6\epsilon, 1\}$. For $x$ with $0 < |x - 3| < \delta$, we have

$$\begin{aligned} |f(x) - L| &= \left|\frac{1}{x} - \frac{1}{3}\right| \\ &= \left|\frac{x-3}{3x}\right| \\ &< \frac{\delta}{6} \\ &< \epsilon. \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, $\lim_{x \to 3} \frac{1}{x} = \frac{1}{3}$.

□

**Exercise 4.2.11** (**Squeeze Theorem**)**.** Let $f$, $g$, and $h$ satisfy $f(x) \leq g(x) \leq h(x)$ for all $x$ in some common domain $A$. If $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} h(x) = L$ at some limit point $c$ of $A$, show that $\lim_{x \to c} g(x) = L$ as well.

**Solution.** *Proof.* Let $f$, $g$, and $h$ satisfy $f(x) \leq g(x) \leq h(x)$ for all $x$ in some common domain $A$. Suppose $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} h(x) = L$ at some limit point $c$ of $A$.

Let $\epsilon > 0$. Since $\lim_{x \to c} f(x) = L$, there exists $\delta_1 > 0$ so that $|f(x) - L| < \epsilon$ whenever $0 < |x - c| < \delta_1$. Since $\lim_{x \to c} h(x) = L$, there exists $\delta_2 > 0$ so that $|h(x) - L| < \epsilon$ whenever $0 < |x - c| < \delta_2$.

Let $\delta = \min\{\delta_1, \delta_2\}$ and let $0 < |x - c| < \delta$. Note that

$$L - \epsilon < f(x) < L + \epsilon$$
$$L - \epsilon < h(x) < L + \epsilon.$$

Using the hypothesis relating $f$, $g$, and $h$, we get

$$L - \epsilon < f(x) \leq g(x) \leq h(x) < L + \epsilon,$$

so we see that $|g(x) - L| < \epsilon$.

Since $\epsilon > 0$ is arbitrary, $\lim_{x \to c} g(x) = L$. □