Homework #6 Solutions
Due Monday, September 22

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Exercise 2.6.2. Give an example of each of the following, or argue that such a request is impossible.

(a) A Cauchy sequence that is not monotone.

(b) A Cauchy sequence with an unbounded subsequence.

(c) A divergent monotone sequence with a Cauchy subsequence.

(d) An unbounded sequence containing a subsequence that is Cauchy.

Solution. (a) The sequence

$$1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \frac{1}{5}, -\frac{1}{6}, \ldots$$

converges to 0, so it is a Cauchy sequence. However, it's evidently not monotone.

(b) This is not possible. Every Cauchy sequence converges and is therefore bounded, so it cannot have an unbounded subsequence.

(c) This is not possible. A monotone sequence is convergent if and only if it is bounded. So, for a monotone sequence to diverge, it must be unbounded. Since the sequence is monotone, any subsequence must likewise be unbounded and therefore cannot be a Cauchy sequence (since Cauchy sequences are bounded).

(d) The sequence

$$1, 1, 1/2, 2, 1/3, 3, 1/4, 4, 1/5, 5, 1/6, 6, \ldots$$

This sequence is unbounded since it contains the subsequence

$$1, 2, 3, 4, 5, 6, \ldots,$$

but contains the convergent subsequence

$$1, 1/2, 1/3, 1/4, 1/5, 1/6, \ldots,$$

which is a Cauchy sequence.

Exercise 2.6.3. If (x_n) and (y_n) are Cauchy sequences, then one easy way to prove that $(x_n + y_n)$ is Cauchy is to use the Cauchy Criterion. By Theorem 2.6.4, (x_n) and (y_n) must be convergent, and the Algebra Limit Theorem then implies $(x_n + y_n)$ is convergent and hence Cauchy.

(a) Give a direct argument that $(x_n + y_n)$ is a Cauchy sequence that does not use the Cauchy Criterion or the Algebraic Limit Theorem.

(b) Do the same for the product $(x_n y_n)$.

Solution. (a) *Proof.* Let (x_n) and (y_n) be Cauchy sequences. Consider the sequence $(x_n + y_n)$.

Let $\epsilon > 0$. Since (x_n) is a Cauchy sequence, there exists N_1 so that $|x_n - x_m| < \epsilon/2$ whenever $n,\ m \geq N_1$. Similarly, since (y_n) is a Cauchy sequence, there exists N_2 so that $|y_n - y_m| < \epsilon/2$ whenever $n,\ m \geq N_2$. Let $N = \max\{N_1, N_2\}$ and let $n,\ m \geq N$. Then

$$\begin{aligned} |(x_n + y_n) - (x_m + y_m)| &= |(x_n - x_m) + (y_n - y_m)| \\ &\leq |x_n - x_m| + |_n - y_m)| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, $(x_n + y_n)$ is a Cauchy sequence. □

(b) *Proof.* Let (x_n) and (y_n) be Cauchy sequences. Consider the sequence $(x_n y_n)$.

By Lemma 2.6.3, since these two sequences are Cauchy, they are bounded. So, there exist $M_x,\ M_y \in \mathbb{R},\ M_x,\ M_y > 0$ so that

$$\begin{aligned} |x_n| &\leq M_x, \\ |y_n| &\leq M_y \end{aligned}$$

for all $n \in \mathbb{N}$.

Since (x_n) is a Cauchy sequence, there exists $N_1 \in \mathbb{N}$ so that

$$|x_n - x_m| < \frac{\epsilon}{2M_y}$$

whenever $n,\ m \geq N_1$.

Since (y_n) is a Cauchy sequence, there exists $N_2 \in \mathbb{N}$ so that

$$|y_n - y_m| < \frac{\epsilon}{2M_x}$$

whenever $n,\ m \geq N_2$.

Let $N = \max\{N_1, N_2\}$ and let n, $m \geq N$. Then

$$\begin{aligned}
|x_n y_n - x_m y_m| &= |x_n y_n - x_m y_n + x_m y_n - x_m y_m| \\
&\leq |x_n - x_m|\,|y_n| + |x_m|\,|y_n - y_m| \\
&< \frac{\epsilon}{2M_y} \cdot M_y + M_x \cdot \frac{\epsilon}{2M_x} = \epsilon.
\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, $(x_n y_n)$ is a Cauchy sequence. $\square$

Exercise 2.6.7. Exercises 2.4.4 and 2.5.4 establish the equivalence of the Axiom of Completeness and the Monotone Convergence Theorem. They also show the Nested Interval Property is equivalent to these other two in the presence of the Archimedean Property.

(a) Assume the Bolzano–Weierstrass Theorem is true and use it to construct a proof of the Monotone Convergence Theorem without making any appeal to the Archimedean Property. This shows that BW, AoC, and MCT are all equivalent.

(b) Use the Cauchy Criterion to prove the Bolzano–Weierstrass Theorem, and find the point in the argument where the Archimedean Property is implicitly required. This establishes the final link in the equivalence of the five characterizations of completeness discussed at the end of Section 2.6.

(c) How do we know it is impossible to prove the Axiom of Completeness starting from the Archimedean Property?

Solution. (a) Suppose the Bolzano-Weierstrass theorem is true: Every bounded sequence contains a convergent subsequence.

Suppose (a_n) is a bounded monotone sequence. By the Bolzano-Weierstrass theorem, (a_n) contains a convergent subsequence (a_{n_k}). Suppose (a_{n_k}) converges to a.

Let $\epsilon > 0$. Since (a_{n_k}) converges to a, there exists $K \in \mathbb{N}$ so that $|a_{n_k} - a| < \epsilon$ for all $k \geq K$. Since (a_{n_k}) is a subsequence of the monotone sequence (a_n), the sequence (a_{n_k}) is also monotone.

Let $N = n_K$ and let $n \geq N$. Since $n \geq n_K$ and (a_{n_k}) is a subsequence of (a_n), there must be some $k \geq K$ so that a_n is between a_{n_K} and a_{n_k}. Since $|a_{n_K} - a| < \epsilon$ and $|a_{n_k} - a| < \epsilon$ and a_n is between a_{n_K} and a_{n_k}, we must have $|a_n - a| < \epsilon$.

This proves that (a_n) likewise converges to a.

Since (a_n) is an arbitrary bounded monotone sequence, the Monotone Convergence Theorem follows.

(b) *Proof.* Let (a_n) be a bounded sequence.

Since (a_n) is bounded, there exists $M \in \mathbb{R}$, $M > 0$, so that $-M \leq a_n \leq M$ for all $n \in \mathbb{N}$. Since $a_n \in [-M, M]$ for all $n \in \mathbb{N}$, either the first half of this interval, $[-M, 0]$, or the second half of this interval, $[0, M]$, must contain a_n for infinitely many $n \in \mathbb{N}$. Call this interval I_1 and let n_1 be the smallest natural number so that $a_{n_1} \in I_1$.

Next, take I_1 and divide it into two equal halves. Since I_1 contains a_n for infinitely many $n \in \mathbb{N}$, one of these halves must contain a_n for infinitely many $n \in \mathbb{N}$. Call this half I_2. Since I_2 contains a_n for infinitely many values of $n \in \mathbb{N}$, we choose a_{n_2} so that $n_2 > n_1$ and $a_{n_2} \in I_2$.

Inductively construct a sequence of closed intervals $I_1 \supseteq I_2 \supseteq I_3 \supseteq I_4 \supseteq \cdots$ with the diameter of $I_k = M/2^{k-1}$ and a subsequence (a_{n_k}) so that $a_{n_k} \in I_k$.

We show that the subsequence (a_{n_k}) is a Cauchy sequence. Let $\epsilon > 0$. Choose $K \in \mathbb{N}$ sufficiently large so that $M/2^{K-1} < \epsilon$ and let $k, \ell \geq K$. Since $k, \ell \geq K$, $a_{n_k} \in I_k \subset I_K$ and $a_{n_\ell} \in I_\ell \subset I_K$. Since I_K has diameter $M/2^{K-1}$,

$$|a_{n_k} - a_{n_\ell}| \leq M/2^{K-1} < \epsilon.$$

Since $\epsilon > 0$ is arbitrary, (a_{n_k}) is a Cauchy sequence, so by the Cauchy Criterion, (a_{n_k}) converges.

Since (a_n) is an arbitrary bounded sequence, this shows every bounded sequence has a convergent subsequence, i.e. the Bolzano-Weierstrass Theorem. □

(c) The Archimedean Property is true for the rational numbers:

Proposition. *Given any number $q \in \mathbb{Q}$, there exists an $n \in \mathbb{N}$ satisfying $n > q$.*

(How do you prove this?)

If the Axiom of Completeness followed from the Archimedean Property, the rational numbers would be complete, which they are not.

Exercise 2.7.1. Proving the Alternating Series Test (Theorem 2.7.7) amounts to showing that the sequence of partial sums

$$s_n = a_1 - a_2 + a_3 - \cdots \pm a_n$$

converges. (The opening example in Section 2.1 includes a typical illustration of (s_n).) Different characterizations of completeness lead to different proofs.

(a) Prove the Alternating Series Test by showing that (s_n) is a Cauchy sequence.

(b) Supply another proof for this result using the Nested Interval Property (Theorem 1.4.1).

(c) Consider the subsequences (s_{2n}) and (s_{2n+1}), and show how the Monotone Convergence Theorem leads to a third proof for the Alternating Series Test.

Solution. (a)

Theorem (Alternating Series Test)**.** *Let (a_n) be a sequence satisfying,*

(a) $a_1 \geq a_2 \geq a_3 \geq \cdots \geq a_n \geq a_{n+1} \geq \cdots$

(b) $(a_n) \to 0$.

Then the alternating series $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ *converges.*

Proof. Let (a_n) be a sequence satisfying,

(a) $a_1 \geq a_2 \geq a_3 \geq \cdots \geq a_n \geq a_{n+1} \geq \cdots$

(b) $(a_n) \to 0$.

Let (s_n) be the sequence of partial sums for the series $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$.

We remark that for $n > m \geq N$ and n and m of the same parity we have

$$s_n - s_m = a_{m+1} - (a_{m+2} - a_{m+3}) - \cdots - (a_{n-2} - a_{n-1}) - a_n \leq a_{m+1}. \tag{1}$$

Let $\epsilon > 0$. Since (a_n) converges to zero, we may choose $N \in \mathbb{N}$ so that $a_n = |a_n| < \epsilon$ whenever $n \geq N$. Let $n > m \geq N$. By Equation (1), $|s_n - s_m| \leq a_{m+1} < \epsilon$.

Hence (s_n) is a Cauchy sequence and therefore converges by the Cauchy Criterion. It follows that the series $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ converges by definition. □

(b)

Theorem (Alternating Series Test)**.** *Let (a_n) be a sequence satisfying,* *(a)* $a_1 \geq a_2 \geq a_3 \geq \cdots \geq a_n \geq a_{n+1} \geq \cdots$

(b) $(a_n) \to 0$.

Then the alternating series $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ *converges.*

Proof. Let (a_n) be a sequence satisfying,

(i) $a_1 \geq a_2 \geq a_3 \geq \cdots \geq a_n \geq a_{n+1} \geq \cdots$

(ii) $(a_n) \to 0$.

Let (s_n) be the sequence of partial sums for the series $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$. We note that $s_{2n} = s_{2n-1} - a_{2n} < s_{2n-1}$. Let $I_n = [s_{2n}, s_{2n-1}]$ for $n \in \mathbb{N}$. Also,

$$s_{2n+2} = s_{2n} + (a_{2n+1} - a_{2n+2}) \geq s_{2n}$$
$$s_{2n+1} = s_{2n-1} - (a_{2n} - a_{2n+1}) \leq s_{2n-1}.$$

So, the intervals I_n are nested:

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq I_4 \supseteq \dots$$

By the Nested Interval Property, $\cap_{n=1}^{\infty} I_n$ is nonempty. Let $x \in \cap_{n=1}^{\infty} I_n$. Since $x \in I_n$ for all $n \in \mathbb{N}$, we have $s_{2n} \leq x \leq s_{2n-1}$ for all $n \in \mathbb{N}$. That is $0 \leq x - s_{2n} \leq s_{2n-1} - s_{2n} = a_{2n}$.

Let $\epsilon > 0$. Since (a_n) converges to zero, there exists $n \in \mathbb{N}$ so that if $n \geq N$ then $0 \leq a_n < \epsilon$. Let $n \geq N$. Then $0 \leq x - s_{2n} \leq s_{2n-1} - s_{2n} = a_{2n} < \epsilon$. Since $\epsilon > 0$ is arbitrary, (s_{2n}) converges to x. But then $(s_{2n-1}) = (s_{2n} + a_{2n})$ also converges to x by the Algebraic Limit Theorem since (a_n) converges to zero. This suffices to prove that (s_n) converges to x, so the series $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ converges by definition. □

(c)

Theorem (Alternating Series Test)**.** *Let* (a_n) *be a sequence satisfying,*

(i) $a_1 \geq a_2 \geq a_3 \geq \cdots \geq a_n \geq a_{n+1} \geq \cdots$

(ii) $(a_n) \to 0$.

Then the alternating series $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ *converges.*

Proof. Let (a_n) be a sequence satisfying,

(i) $a_1 \geq a_2 \geq a_3 \geq \cdots \geq a_n \geq a_{n+1} \geq \cdots$

(ii) $(a_n) \to 0$.

Let (s_n) be the sequence of partial sums for the series $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$. We note that

$$\begin{aligned} s_{2n+2} &= s_{2n} + (a_{2n+1} - a_{2n+2}) \geq s_{2n} \\ s_{2n+1} &= s_{2n-1} - (a_{2n} - a_{2n+1}) \leq s_{2n-1}. \end{aligned}$$

Further

$$\begin{aligned} s_{2n-1} &= (a_1 - a_2) + (a_3 - a_4) + \cdots + (a_{2n-3} - a_{2n-2}) + a_{2n-1} \\ &\geq a_1 - a_2 \\ s_{2n} &= a_1 - (a_2 - a_3) - \cdots - (a_{2n-2} - a_{2n-1}) - a_{2n} \\ &\leq a_1 \end{aligned}$$

So, (s_{2n}) is monotone increasing and bounded above and (s_{2n-1}) is monotone decreasing and bounded below. By the Monotone Convergence Theorem, (s_{2n}) converges, say to x, and (s_{2n-1}) converges, say to y. By the Algebraic Limit Theorem, $(a_{2n}) = (s_{2n} - s_{2n-1})$ converges to $x - y$, but (a_{2n}) is a subsequence of (a_n), which converges to zero, so (a_{2n}) must converge to zero. Hence $x - y = 0$, i.e. $x = y$, and it follows that (s_n) converges to x. □

Exercise 2.7.3. (a) Provide the details for the proof of the Comparison Test (Theorem 2.7.4) using the Cauchy Criterion for Series.

(b) Give another proof for the Comparison Test, this time using the Monotone Convergence Theorem.

Solution.

Theorem (The Comparison Test)**.** *Assume* (a_k) *and* (b_k) *are sequences satisfying* $0 \le a_k \le b_k$ *for all* $k \in \mathbb{N}$.

(i) If $\sum_{k=1}^{\infty} b_k$ *converges, then* $\sum_{k=1}^{\infty} a_k$ *converges.*

(ii) If $\sum_{k=1}^{\infty} a_k$ *diverges, then* $\sum_{k=1}^{\infty} b_k$ *diverges.*

(i) *Proof.* Let (a_k) and (b_k) be sequences satisfying $0 \le a_k \le b_k$ for all $k \in \mathbb{N}$. We first note that (ii) is the contrapositive of (i), so it suffices to prove (i).

Suppose $\sum_{k=1}^{\infty} b_k$ converges. Let $\epsilon > 0$. Since $\sum_{k=1}^{\infty} b_k$ converges, by the Cauchy Criterion for Series, there exists $N \in \mathbb{N}$ so that whenever $n > m \ge N$ it follows that

$$|b_{m+1} + b_{m+2} + \cdots + b_n| < \epsilon.$$

Since $0 \le a_k \le b_k$ for all $k \in \mathbb{N}$,

$$0 \le a_{m+1} + a_{m+2} + \cdots + a_n \le b_{m+1} + a_{m+2} + \cdots + b_n$$

whenever n, $m \in \mathbb{N}$ with $n > m$. Hence

$$|a_{m+1} + a_{m+2} + \cdots + a_n| \le |b_{m+1} + a_{m+2} + \cdots + b_n| < \epsilon.$$

By the Cauchy Criterion for Series, $\sum_{k=1}^{\infty} a_k$ converges. □

(ii) *Proof.* Let (a_n) and (b_n) be sequences satisfying $0 \le a_k \le b_k$ for all $k \in \mathbb{N}$. We first note that (ii) is the contrapositive of (i), so it suffices to prove (i).

Let $s_n = a_1 + a_2 + \cdots + a_n$ and let $t_n = b_1 + b_2 + \cdots + b_n$. For $n \in \mathbb{N}$, we have

$$s_n = a_1 + \cdots + a_n \le b_1 + \cdots + b_n = t_n.$$

We remark that since the terms a_n and b_n are nonnegative, the sequences (s_n) and (t_n) are monotone increasing.

Suppose $\sum_{k=1}^{\infty} b_k$ converges. This means that the sequence (t_n) converges to some $t \in \mathbb{R}$. Since t_n is monotone increasing, we have $t_n \le t$ for all $n \in \mathbb{N}$. Then we have

$$s_n \le t_n \le t.$$

From this, we see that the sequence (s_n) is bounded. Since (s_n) is a monotone, bounded sequence, it converges by the Monotone Convergence Theorem. But this means the series $\sum_{k=1}^{\infty} a_k$ converges. □

Exercise 2.7.4. Give an example of each or explain why the request is impossible referencing the proper theorem(s).

(a) Two series $\sum x_n$ and $\sum y_n$ that both diverge but where $\sum x_n y_n$ converges.

(b) A convergent series $\sum x_n$ and a bounded sequence (y_n) such that $\sum x_n y_n$ diverges.

(c) Two sequences (x_n) and (y_n) where $\sum x_n$ and $\sum(x_n + y_n)$ both converge but $\sum y_n$ diverges.

(d) A sequence (x_n) satisfying $0 \leq x_n \leq 1/n$ where $\sum(-1)^n x_n$ diverges.

Examples. (a) Let $x_n = y_n = 1/n$ for all $n \in \mathbb{N}$. Then $\sum_{n=1}^{\infty} x_n = \sum_{n=1}^{\infty} y_n$ diverges since it is the harmonic series. However, $x_n y_n = 1/n^2$, and $\sum_{n=1}^{\infty} x_n y_n = \sum_{n=1}^{\infty} 1/n^2$ converges (since it's a p-series with $p > 1$).

(b) Let $x_n = (-1)^{n-1}/n$ and $y_n = (-1)^{n-1}$. Then $\sum x_n$ is the alternating harmonic series, which converges. The sequence (y_n) is bounded and $\sum x_n y_n$ is the harmonic series, so it diverges.

(c) This is not possible by the Algebraic Limit Theorem for series. If $\sum(x_n + y_n)$ and $\sum x_n$ converge, the series formed by their term by term difference, $\sum y_n$, must also converge.

(d) Let

$$x_n = \begin{cases} 0 & \text{when } n \text{ is odd} \\ \dfrac{1}{n} & \text{when } n \text{ is even} \end{cases}$$

Then (x_n) is the sequence $(0, \frac{1}{2}, 0, \frac{1}{4}, 0, \frac{1}{6}, \dots)$. Then $0 \leq x_n \leq 1/n$ and

$$\sum(-1)^n x_n = \sum \frac{1}{2n} = \frac{1}{2} \sum \frac{1}{n},$$

one-half the harmonic series, with zeroes scattered through it. This series diverges.[1]

[1] Shamelessly stolen from Will Blevins.