Homework #5
Due Monday, September 15

William M. Faucette

Exercise 2.4.1. (a) Prove that the sequence defined by $x_1 = 3$ and

$$x_{n+1} = \frac{1}{4 - x_n}$$

converges.

(b) Now that we know $\lim x_n$ exists, explain why $\lim x_{n+1}$ must also exist and equal the same value.

(c) Take the limit of each side of the recursive equation in part (a) of this exercise to explicitly compute $\lim x_n$.

Solution. (a) *Proof.* Let (x_n) be the sequence defined recursively by $x_1 = 3$ and

$$x_{n+1} = \frac{1}{4 - x_n}$$

for $n \in \mathbb{N}$.

We prove that

$$\frac{1}{4} < x_{n+1} \leq x_n$$

for all n using the Principle of Mathematical Induction.

For $n = 1$, we have $x_1 = 3$ and $x_2 = 1$, so the result is true for $n = 1$. Suppose the result is true for some $n \in \mathbb{N}$. That is, suppose

$$\frac{1}{4} < x_{n+1} \leq x_n$$

for some $n \in \mathbb{N}$. Then

$$\begin{aligned}
0 < \frac{1}{4} &< x_{n+1} \leq x_n \\
4 &> 4 - x_{n+1} \geq 4 - x_n \\
\frac{1}{4} &< \frac{1}{4 - x_{n+1}} \leq \frac{1}{4 - x_n} \\
\frac{1}{4} &< x_{n+2} \leq x_{n+1}.
\end{aligned}$$

So, the result is true for $n+1$. By the Principle of Mathematical Induction, this proves that

$$\frac{1}{4} < x_{n+1} \leq x_n$$

for all $n \in \mathbb{N}$.

This shows that the sequence (x_n) is decreasing and bounded below. Therefore by the Monotone Convergence Theorem, the sequence (x_n) converges. □

(b) Since (x_n) converges, (x_{n+1}) is simply the same sequence starting with the second term. So, it converges and converges to the same value.

For a rigorous argument, suppose (x_n) converges to L. Let $\epsilon > 0$. Since (x_n) converges to L, there exists $N \in \mathbb{N}$ so that $|x_n - L| < \epsilon$ whenever $n \geq N$. Let $n \geq N$. Then $|x_{n+1} - L| < \epsilon$ since $n + 1 > n \geq N$. Thus, (x_{n+1}) converges to L as well.

(c) Suppose (x_n) converges to x. We find the value of x by taking the limit of both sides of the recursion relation

$$x_{n+1} = \frac{1}{4 - x_n}.$$

Taking limits, we get

$$\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} \frac{1}{4 - x_n}$$
$$x = \frac{1}{4 - x}.$$

Solving this last equation, we get $x = 2 \pm \sqrt{3}$. Since $x_n \leq 3$ for all values of n, (x_n) cannot converge to $2 + \sqrt{3} \approx 3.732$. So, we see that $x = 2 - \sqrt{3}$.

Exercise 2.5.1. Give an example of each of the following, or argue that such a request is impossible.

(a) A sequence that has a subsequence that is bounded but contains no subsequence that converges.

(b) A sequence that does not contain 0 or 1 as a term but contains subsequences converging to each of these values.

(c) A sequence that contains subsequences converging to every point in the infinite set $\{1, 1/2, 1/3, 1/4, 1/5, \dots\}$.

(d) A sequence that contains subsequences converging to every point in the infinite set $\{1, 1/2, 1/3, 1/4, 1/5, \dots\}$, and no subsequences converging to points outside of this set.

Solution. (a) This is not possible. Suppose a sequence has a subsequence that is bounded. By the Bolzano-Weierstrass Theorem, the bounded subsequence itself has a convergent subsequence. But this convergent subsequence is likewise a subsequence of the original sequence. So, the original sequence must contain a subsequence that converges.

(b) Consider the sequence given by

$$a_n = \begin{cases} \dfrac{1}{n+1} & \text{if } n \text{ is odd} \\ 1 - \dfrac{1}{n} & \text{if } n \text{ is even.} \end{cases}$$

The even terms converge to 1:

$$\left(\frac{1}{2}, \frac{3}{4}, \frac{5}{6}, \frac{7}{8}, \frac{9}{10}, \dots \right).$$

The odd terms converge to 0:

$$\left(\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \frac{1}{10}, \dots \right).$$

Notice that the original sequence does not contain 0 or 1.

(c) Consider the sequence

$$\left(1, 1, \frac{1}{2}, 1, \frac{1}{2}, \frac{1}{3}, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \dots \right)$$

This sequence has subsequences converging to $1/n$ for each $n \in \mathbb{N}$.

(d) An example here is not possible. Any sequence having a subsequence converging to each element of the set

$$\{1, 1/2, 1/3, 1/4, 1/5, \dots\},$$

must also contain a subsequence converging to 0, which is not in this set.

Suppose (b_n) is a sequence which contains subsequences converging to each element of the set

$$\{1, 1/2, 1/3, 1/4, 1/5, \dots\},$$

Let $\epsilon > 0$ be arbitrary. We can find a subsequence $(b^n_{n_k})$ converging to $1/n$. Choose N_1 so that $|b^1_{N_1} - 1| < \epsilon/2$. Choose $N_2 > N_1$ so that $|b^2_{N_2} - 1/2| < \epsilon/2$. Continuing in this way, choose $N_{k+1} > N_k$ so that $|b^k_{N_k} - 1/k| < \epsilon/2$.

Consider the subsequence (c_k) defined by $c_k = b^k_{N_k}$. Pick N large enough so that $1/N < \epsilon/2$. Then for $n \geq N$ we have

$$|c_n| = \left| c_n - \frac{1}{n} + \frac{1}{n} \right| \leq \left| c_n - \frac{1}{n} \right| + \left| \frac{1}{n} \right| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

So, we see the subsequence (c_k) converges to 0.[1]

[1] This proof is adapted from a shamelessly stolen proof outline by Ulisse Mini and Jesse Li.

Exercise 2.5.5. Assume (a_n) is a bounded sequence with the property that every convergent subsequence of (a_n) converges to the same limit $a \in \mathbb{R}$. Show that (a_n) must converge to a.

Solution. *Proof.* Suppose (a_n) does not converge to a. Then there exists $\epsilon > 0$ so that for every $N \in \mathbb{N}$ there exists $n \geq N$ so that $|a_n - a| \geq \epsilon$. For $N = 1$, choose $n_1 \in \mathbb{N}$, $n_1 \geq N$, so that $|a_{n_1} - a| \geq \epsilon$. Next, for $N = n_1 + 1$, choose $n_2 \in \mathbb{N}$, $n_2 \geq N$, so that $|a_{n_2} - a| \geq \epsilon$. Note that $n_2 > n_1$.

Having constructed $a_{n_1}, a_{n_2}, \ldots, a_{n_k}$ with $n_1 < \cdots < n_k$ and $|a_{n_i} - a| \geq \epsilon$ for all i, $1 \leq i \leq k$, for $N = n_k + 1$, choose $n_{k+1} \in \mathbb{N}$, $n_{k+1} \geq N$, so that $|a_{n_{k+1}} - a| \geq \epsilon$. Note that $n_1 < \cdots < n_k < n_{k+1}$.

Now the sequence (a_{n_k}), being a subsequence of (a_n), is bounded, so by the Bolzano-Weierstrass Theorem, it has a convergent subsequence. But $|a_{n_k} - a| \geq \epsilon$ for all k, so this convergent subsequence can't possibly converge to a. This contradicts the hypothesis. $\square$