Homework #3 Solutions
Due Wednesday, September 3

William M. Faucette

Exercise 1.6.10. As a final exercise, answer each of the following by establishing 1–1 correspondence with a set of known cardinality.

(a) Is the set of all functions from $\{0,1\}$ to $\mathbb{N}$ countable or uncountable?

(b) is the set of all functions from $\mathbb{N}$ to $\{0,1\}$ countable or uncountable?

Solution. (a) The set of all functions S from $\{0,1\}$ to $\mathbb{N}$ countable. Define a function

$$\Phi : S \to \mathbb{N} \times \mathbb{N}$$

by $\Phi(f) = (f(0), f(1))$. This function is easily seen to be bijective.

Now the set $\mathbb{N} \times \mathbb{N}$ is countable as follows. Define a function

$$g : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$$

by $g(m,n) = 2^m 3^n$. By the Fundamental Theorem of Arithmetic, this gives a bijection between $\mathbb{N} \times \mathbb{N}$ and an infinite subset of $\mathbb{N}$. By Theorem 1.4.12, this subset must be countable, so $\mathbb{N} \times \mathbb{N}$, and therefore S, is countable.

(b) *Proof.* The set of all functions from $\mathbb{N}$ to $\{0,1\}$ uncountable. We define a function Φ from the power set of $\mathbb{N}$, $\mathscr{P}(\mathbb{N})$, to the set S of all functions from $\mathbb{N}$ to $\{0,1\}$ as follows.

For $U \subseteq \mathbb{N}$, define $\mathbf{a}(U) = (a_1, a_2, a_3, \dots) \in S$ as follows. Let

$$a_n = \begin{cases} 1 \text{ if } n \in U \\ 0 \text{ if } n \notin U \end{cases}$$

Define $\Phi : \mathscr{P}(\mathbb{N}) \to S$ by $\Phi(U) = \mathbf{a}(U)$.

This gives a bijection between $\mathscr{P}(\mathbb{N})$ and S. However, we know from the text that $\mathscr{P}(\mathbb{N})$ is uncountable, so the set of all functions from $\mathbb{N}$ to $\{0,1\}$ is likewise uncountable. □

Exercise 2.2.2. Verify, using the definition of convergence of a sequence, that the following sequences converge to the proposed limit.

(a) $\lim_{n\to\infty} \frac{2n+1}{5n+4} = \frac{2}{5}$.

(b) $\lim_{n\to\infty} \frac{2n^2}{n^3+3} = 0$.

(c) $\lim_{n\to\infty} \frac{\sin(n^2)}{\sqrt[3]{n}} = 0$.

Solution. (a) *Proof.* Let $\epsilon > 0$. Using the Archimedean Property, choose $N \in \mathbb{N}$ so that $N > 3/(25\epsilon)$. Let $n \geq N$. Then

$$\left|\frac{2n+1}{5n+4} - \frac{2}{5}\right| = \left|\frac{3}{20+25n}\right| < \left|\frac{3}{25n}\right| \leq \left|\frac{3}{25N}\right| = \frac{3}{25} \cdot \frac{1}{N} < \frac{3}{25} \cdot \frac{25}{3}\epsilon = \epsilon.$$

Since $\epsilon > 0$ is arbitrary,

$$\lim_{n\to\infty} \frac{2n+1}{5n+4} = \frac{2}{5},$$

by definition. □

(b) *Proof.* Let $\epsilon > 0$. Using the Archimedean Property, choose $N \in \mathbb{N}$ so that $N > 2/\epsilon$. Let $n \geq N$. Then

$$\left|\frac{2n^2}{n^3+3} - 0\right| = \frac{2n^2}{n^3+3} < \frac{2n^2}{n^3} = \frac{2}{n} \leq \frac{2}{N} < \epsilon.$$

Since $\epsilon > 0$ is arbitrary,

$$\lim_{n\to\infty} \frac{2n^2}{n^3+3} = 0,$$

by definition. □

(c) *Proof.* Let $\epsilon > 0$. Using the Archimedean Property, choose $N \in \mathbb{N}$ so that $N > 1/\epsilon^3$. Let $n \geq N$. Then

$$\left|\frac{\sin(n^2)}{\sqrt[3]{n}}\right| \leq \frac{1}{\sqrt[3]{n}} \leq \frac{1}{\sqrt[3]{N}} < \epsilon.$$

Since $\epsilon > 0$ is arbitrary,

$$\lim_{n\to\infty} \frac{\sin(n^2)}{\sqrt[3]{n}} = 0,$$

by definition.

□

Exercise 2.2.4. Give an example of each or state that the request is impossible. For any that are impossible, give a compelling argument for why that is the case.

(a) A sequence with an infinite number of ones that does not converge to one.

(b) A sequence with an infinite number of ones that converges to a limit not equal to one.

(c) A divergent sequence such that for every $n \in \mathbb{N}$ it is possible to find n consecutive ones somewhere in the sequence.

Solution. (a) The sequence

$$0, 1, 0, 1, 0, 1, 0, 1, \ldots$$

has an infinite number of ones that but does not converge to 1.

(b) Any sequence with an infinite number of 1's has a subsequence that converges to 1, namely the constant subsequence consisting of the entries in the sequence that equal 1. If in addition, the sequence itself converges, then the sequence must converge to 1, by Theorem 2.5.2. So, it is not possible to find a sequence satisfying (b).

(c) The sequence

$$1, 0, 1, 1, 0, 1, 1, 1, 0, 1, 1, 1, 1, 0, \ldots$$

contains n consecutive 1's for every $n \in \mathbb{N}$. However, it diverges by Theorem 2.5.2, since it has subsequences converging to 0 and 1.

Exercise 2.2.6. Prove Theorem 2.2.7:

Theorem. *The limit of a sequence, when it exists, must be unique.*

To get started, assume $(a_n) \to a$ and also that $(a_n) \to b$. Now argue that $a = b$.

Solution 1. *Proof.* Let (a_n) be a sequence and suppose (a_n) converges to both a and b.

Let $\epsilon > 0$. Since (a_n) converges to a, there exists $N_1 \in \mathbb{N}$ so that

$$|a_n - a| < \frac{\epsilon}{2}.$$

whenever $n \geq N_1$.

Since (a_n) converges to b, there exists $N_2 \in \mathbb{N}$ so that

$$|a_n - b| < \frac{\epsilon}{2}.$$

whenever $n \geq N_2$.

Let $n \geq \max\{N_1, N_2\}$. Then

$$|a - b| = |a - a_n + a_n - b| \leq |a - a_n| + |a_n - b| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Since $\epsilon > 0$ is arbitrary, $|a - b| < \epsilon$ for all $\epsilon > 0$. This implies $|a - b| = 0$, so $a = b$. □

Solution 2 (Several students (edited))**.**

Proof. Let (a_n) be a sequence and suppose (a_n) converges to both a and b.

Assume $a \neq b$. Let $\epsilon = |a - b| > 0$.

Since (a_n) converges to a, there exists $N_1 \in \mathbb{N}$ so that $|a_n - a| < \epsilon/2$ for $n \geq N_1$. Since (a_n) converges to b, there exists $N_2 \in \mathbb{N}$ so that $|a_n - b| < \epsilon/2$ for $n \geq N_2$.

Let $N = \max\{N_1, N_2\}$ and let $n \geq N$. Then

$$|a - b| = |a - a_n + a_n - b| \leq |a - a_n| + |a_n - b| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon = |a - b|.$$

This is a contradiction.

So, $a = b$. □

Exercise 2.2.7. Here are two useful definitions:

(i) A sequence (a_n) is *eventually* in a set $A \subseteq \mathbb{R}$ if there exists an $N \in \mathbb{N}$ such that $a_n \in A$ for all $n \geq N$.

(ii) A sequence (a_n) is *frequently* in a set $A \subseteq \mathbb{R}$ if, for every $N \in \mathbb{N}$, there exists an $n \geq N$ such that $a_n \in A$.

(a) Is the sequence $(-1)^n$ eventually or frequently in the set $\{1\}$?

(b) Which definition is stronger? Does frequently imply eventually or does eventually imply frequently?

(c) Give an alternate rephrasing of Definition 2.2.3B using either frequently or eventually. Which is the term we want?

(d) Suppose an infinite number of terms of a sequence (x_n) are equal to 2. Is (x_n) necessarily eventually in the interval $(1.9, 2.1)$? Is it frequently in $(1.9, 2.1)$?

Solution. (a) The sequence $(-1)^n$ is frequently in the set $\{1\}$ since for every even natural number n, $(-1)^n = 1$. So, $(-1)^n = 1$ for infinitely many values of n.

(b) A sequence being eventually in a set $A \subseteq R$ is stronger than a sequence being frequently in a set $A \subseteq R$.

(c)

Definition. A sequence (a_n) converges to a if, given any ϵ-neighborhood $V_\epsilon(a)$ of a, the sequence (a_n) is eventually in $V_\epsilon(a)$.

(d) Suppose an infinite number of terms of a sequence (x_n) are equal to 2.

The sequence (x_n) is not necessarily eventually in $(1.9, 2.1)$. As an example, take the sequence

$$0, 2, 0, 2, 0, 2, 0, 2, \ldots.$$

This sequence has an infinite number of 2's, but is not eventually in $(1.9, 2.1)$, since for any $N \in \mathbb{N}$, there exists $n \geq N$ so that $a_n = 0$, which is outside the interval.

On the other hand, such a sequence is frequently in the interval $(1.9, 2.1)$. Let $N \in \mathbb{N}$. Since an infinite number of terms of the sequence are equal to 2 and there are finitely many indices between 1 and $N - 1$, there must exist $n \geq N$ with $x_n = 2$. So, the sequence is frequently in $(1.9, 2.1)$.