Homework #1 Solutions
Due Monday, August 18

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Exercise 1.2.5 (De Morgan's Laws). Let A and B be subsets of $\mathbb{R}$.

(a) If $x \in (A \cap B)^c$, explain why $x \in A^c \cup B^c$. This should show that $(A \cap B)^c \subseteq A^c \cup B^c$.

(b) Prove the reverse inclusion $(A \cap B)^c \supseteq A^c \cup B^c$, and conclude that $(A \cap B)^c = A^c \cup B^c$.

Solution. (a) *Proof.* Let $x \in (A \cap B)^c$. Then $x \notin A \cap B$.

Now, $A \cap B$ consists of all those elements y such that $y \in A$ and $y \in B$. Carefully negating this, we see that since $x \notin A \cap B$, either $x \notin A$ or $x \notin B$. Thus, $x \in A^c$ or $x \in B^c$, whereby $x \in A^c \cup B^c$.

Since $x \in (A \cap B)^c$ is arbitrary, this proves $(A \cap B)^c \subseteq A^c \cup B^c$. □

(b) *Proof.* Now suppose $x \in A^c \cup B^c$. Then $x \in A^c$ or $x \in B^c$, so that $x \notin A$ or $x \notin B$. However, this means that $x \notin A \cap B$. So, $x \in (A \cap B)^c$.

Since $x \in A^c \cup B^c$ is arbitrary, this proves $(A \cap B)^c \supseteq A^c \cup B^c$.

Putting parts (a) and (b) together gives us

$$(A \cap B)^c = A^c \cup B^c.$$

□

Exercise 1.2.7. Given a function f and a subset A of its domain, let $f(A)$ represent the range of f over the set A; that is, $f(A) = \{f(x) : x \in A\}$.

(a) Let $f(x) = x^2$. If $A = [0, 2]$ (the closed interval $\{x \in \mathbb{R} : 0 \leq x \leq 2\}$) and $B = [1, 4]$, find $f(A)$ and $f(B)$. Does $f(A \cap B) = f(A) \cap f(B)$ in this case? Does $f(A \cup B) = f(A) \cup f(B)$?

(b) Find two sets A and B for which $f(A \cap B) \neq f(A) \cap f(B)$.

(c) Show that, for an arbitrary function $g : \mathbb{R} \to \mathbb{R}$, it is always true that $g(A \cap B) \subseteq g(A) \cap g(B)$ for all sets A, $B \subseteq \mathbb{R}$.

(d) Form and prove a conjecture about the relationship between $g(A \cup B)$ and $g(A) \cup g(B)$ for an arbitrary function g.

Solution. (a) Let $f(x) = x^2$. Let $A = [0, 2]$ and $B = [1, 4]$. Then $f(A) = [0, 4]$ and $f(B) = [1, 16]$, so $f(A) \cap f(B) = [1, 4]$. Further computing, $A \cap B = [1, 2]$ and $f(A \cap B) = [1, 4]$ So, we see in this case

$$f(A \cap B) = f(A) \cap f(B).$$

Now, we move to the union. We have $A \cup B = [0, 4]$ and $f(A \cup B) = [0, 16]$. From above, $f(A) = [0, 4]$ and $f(B) = [1, 16]$, so $f(A) \cup f(B) = [0, 16]$. So, we see in this case

$$f(A \cup B) = f(A) \cup f(B).$$

(b) Let $A = [1, 2]$ and $B = [-2, -1]$. Then $A \cap B = \emptyset$, so $f(A \cap B) = \emptyset$. On the other hand, $f(A) = f(B) = [1, 4]$, so $f(A) \cap f(B) = [1, 4]$. So, we see that $f(A \cap B) \subsetneq f(A) \cap f(B)$.

(c) *Proof.* Let $g : \mathbb{R} \to \mathbb{R}$ and let A, $B \subseteq \mathbb{R}$. Since $A \cap B \subseteq A$, we have $g(A \cap B) \subseteq g(A)$. Since $A \cap B \subseteq B$, we have $g(A \cap B) \subseteq g(B)$. Since $g(A \cap B)$ is a subset of both $g(A)$ and $g(B)$, $g(A \cap B) \subseteq g(A) \cap g(B)$. (*Proof due to Trystyn Hovey.*) □

(d) **Conjecture:** For an arbitrary function $g : \mathbb{R} \to \mathbb{R}$, it is always true that $g(A) \cup g(B) = g(A \cup B)$.

Proof. Let A, B be subsets of $\mathbb{R}$ and let $g : \mathbb{R} \to \mathbb{R}$.

Since $A \subseteq A \cup B$, we have $g(A) \subseteq g(A \cup B)$. Similarly, since $B \subseteq A \cup B$, we have $g(B) \subseteq g(A \cup B)$. It follows that $g(A) \cup g(B) \subseteq g(A \cup B)$.

For the reverse inclusion, let $y \in g(A \cup B)$. Then there exists $x \in A \cup B$ so that $g(x) = y$. Since $x \in A \cup B$, either $x \in A$ or $x \in B$. If $x \in A$, then $y = g(x) \in g(A) \subseteq g(A) \cup g(B)$. Likewise, if $x \in B$, then $y = g(x) \in g(B) \subseteq g(A) \cup g(B)$. Since $y \in g(A \cup B)$ is arbitrary, $g(A \cup B) \subseteq g(A) \cup g(B)$. These two inclusions prove the result. □

Exercise 1.2.9. Given a function $f : D \to \mathbb{R}$ and a subset $B \subseteq \mathbb{R}$, let $f^{-1}(B)$ be the set of all points from the domain D that get mapped into B; that is,

$$f^{-1}(B) = \{x \in D \mid f(x) \in B\}$$

This set is called the **preimage** of B.

(a) Let $f(x) = x^2$. If A is the closed interval $[0, 4]$ and B is the closed interval $[-1, 1]$, find $f^{-1}(A)$ and $f^{-1}(B)$. Does $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$ in this case? Does $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$?

(b) The good behavior of preimages demonstrated in (a) is completely general. Show that for an arbitrary function $g : \mathbb{R} \to \mathbb{R}$, it is always true that $g^{-1}(A \cap B) = g^{-1}(A) \cap g^{-1}(B)$ and $g^{-1}(A \cup B) = g^{-1}(A) \cup g^{-1}(B)$ for all sets A, $B \subseteq \mathbb{R}$.

Solution. (a) Let $f(x) = x^2$. Let A be the closed interval $[0, 4]$ and B is the closed interval $[-1, 1]$. The set $f^{-1}(A)$ is the set of all x's that map onto the y-axis between $y = 0$ and $y = 4$. You get this by projecting the graph of $y = x^2$ between $y = 0$ and $y = 4$ onto the x-axis. This gives you the interval $[-2, 2]$.

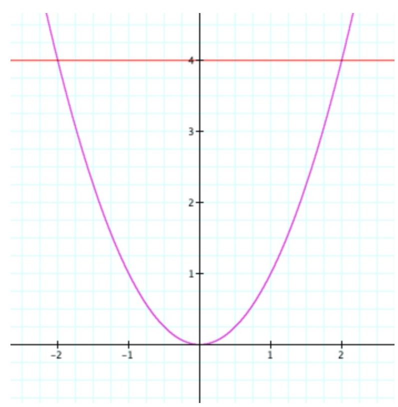

Figure 1: Graph of $y = x^2$

Similarly, the set $f^{-1}(B)$ is the set of all x's that map onto the y-axis between $y = -1$ and $y = 1$. You get this by projecting the graph of $y = x^2$ between $y = -1$ and $y = 1$ onto the x-axis. This gives you the interval $[-1, 1]$.

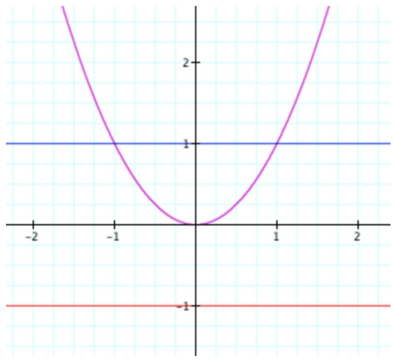

Figure 2: Graph of $y = x^2$

So,

$$f^{-1}(A) = [-2, 2] \text{ and } f^{-1}(B) = [-1, 1].$$

Hence

$$f^{-1}(A) \cap f^{-1}(B) = [-1, 1].$$

We find that $A \cap B = [0, 1]$ and (using the same technique) we have that $f^{-1}(A \cap B) = [-1, 1]$.

So, in this case, we have

$$f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B).$$

(b) *Proof.* Let $g : \mathbb{R} \to \mathbb{R}$ and let $A,\ B \subseteq \mathbb{R}$.

Let $x \in g^{-1}(A \cap B)$. Then $g(x) \in A \cap B$, so that $g(x) \in A$ and $g(x) \in B$. But then $x \in g^{-1}(A)$ and $x \in g^{-1}(B)$. So, $x \in g^{-1}(A) \cap g^{-1}(B)$. Since $x \in g^{-1}(A \cap B)$ is arbitrary, this shows

$$g^{-1}(A \cap B) \subseteq g^{-1}(A) \cap g^{-1}(B).$$

Let $x \in g^{-1}(A) \cap g^{-1}(B)$. Then $x \in g^{-1}(A)$ and $x \in g^{-1}(B)$, whereby $g(x) \in A$ and $g(x) \in B$. Hence $g(x) \in A \cap B$. But this says, $x \in g^{-1}(A \cap B)$. Since $x \in g^{-1}(A) \cap g^{-1}(B)$ is arbitrary, this shows

$$g^{-1}(A) \cap g^{-1}(B) \subseteq g^{-1}(A \cap B).$$

These two inclusions show that

$$g^{-1}(A \cap B) = g^{-1}(A) \cap g^{-1}(B).$$

□

Exercise 1.2.12. Let $y_1 = 6$, and for each $n \in \mathbb{N}$ define $y_{n+1} = (2y_n - 6)/3$.

(a) Use induction to prove that the sequence satisfies $y_n > -6$ for all $n \in \mathbb{N}$.

(b) User another induction argument to show the sequence $(y_1, y_2, y_3, \dots)$ is decreasing.

Solution. (a) *Proof.* First, we have that $y_1 = 6 > -6$, so the result is true for $n = 1$.

Suppose the $y_k > -6$ for some $k \in \mathbb{N}$.

Then

$$\begin{aligned} y_{k+1} &= \frac{2y_k - 6}{3} \\ &= \frac{2}{3}y_k - 2 \\ &> \frac{2}{3}(-6) - 2 \\ &> -6. \end{aligned}$$

By the Principle of Mathematical Induction, $y_n > -6$ for all $n \in \mathbb{N}$.

□

(b) *Proof.* It is really not necessary to use an induction argument here. We note that from (a), we have $\frac{1}{3}y_n > -2$, so we have

$$y_n = \frac{2}{3}y_n + \frac{1}{3}y_n > \frac{2}{3}y_n - 2 = \frac{2y_n - 6}{3} = y_{n+1}.$$

So, the sequence $(y_1, y_2, y_3, \dots)$ is decreasing.

□

Exercise 1.2.13. For this exercise, assume Exercise 1.2.5 has been successfully completed.

(a) Show how induction can be used to conclude that

$$(A_1 \cup A_2 \cup \cdots \cup A_n)^c = A_1^c \cap A_2^c \cap \cdots \cap A_n^c$$

for any finite $n \in \mathbb{N}$.

(b) Explain why induction *cannot* be used to conclude

$$\left(\bigcup_{n=1}^{\infty} A_n \right)^c = \bigcap_{n=1}^{\infty} A_n^c.$$

It might be useful to consider part (a) of Exercise 1.2.3.

(c) Is the statement in part (b) valid? If so, write a proof that does not use induction.

Solution. (a) *Proof.* Let $\{A_i\}$, $i \in \mathbb{N}$, be an indexed family of sets. The result is clearly true for $n = 1$. By Exercise 1.2.3(c), the result is true for $n = 2$.

Suppose the result is true for $n = k$. That is, suppose

$$(A_1 \cup A_2 \cup \cdots \cup A_k)^c = A_1^c \cap A_2^c \cap \cdots \cap A_k^c.$$

Then

$$\begin{aligned}(A_1 \cup A_2 \cup \cdots \cup A_k \cup A_{k+1})^c &= [(A_1 \cup A_2 \cup \cdots \cup A_k) \cup A_{k+1}]^c \\ &= (A_1 \cup A_2 \cup \cdots \cup A_k)^c \cap A_{k+1}^c, \text{ by Exercise 1.2.3(c)} \\ &= (A_1^c \cap A_2^c \cap \cdots \cap A_k^c) \cap A_{k+1}^c, \text{ by the inductive hypothesis} \\ &= A_1^c \cap A_2^c \cap \cdots \cap A_k^c \cap A_{k+1}^c.\end{aligned}$$

This shows the result holds for $n = k+1$. By the Principle of Mathematical Induction, the result holds for all $n \in \mathbb{N}$. □

(b) The Principle of Mathematical Induction cannot be used to prove this result because that principle proves that a result holds for every *finite* $n \in \mathbb{N}$, in this case, every *finite* union or intersection.

(c) **Conjecture:** For an indexed family of sets $\{A_i\}_{i \in \mathbb{N}}$,

$$\left(\bigcup_{n=1}^{\infty} A_n \right)^c = \bigcap_{n=1}^{\infty} A_n^c.$$

Proof. Let $x \in (\bigcup_{n=1}^{\infty} A_n)^c$. Then $x \notin \bigcup_{n=1}^{\infty} A_n$, which means $x \notin A_n$ for all $n \in \mathbb{N}$. So, $x \in A_n^c$ for all $n \in \mathbb{N}$, whereby $x \in \bigcap_{n=1}^{\infty} A_n^c$. Since $x \in (\bigcup_{n=1}^{\infty} A_n)^c$ is arbitrary, $(\bigcup_{n=1}^{\infty} A_n)^c \subseteq \bigcap_{n=1}^{\infty} A_n^c$.

Let $x \in \bigcap_{n=1}^{\infty} A_n^c$. Then $x \in A_n^c$ for all $n \in \mathbb{N}$, so $x \notin A_n$ for any $n \in \mathbb{N}$. Thus $x \notin \bigcup_{n=1}^{\infty} A_n$, whereby $x \in (\bigcup_{n=1}^{\infty} A_n)^c$. Since $x \in \bigcap_{n=1}^{\infty} A_n^c$ is arbitrary, $\bigcap_{n=1}^{\infty} A_n^c \subseteq (\bigcup_{n=1}^{\infty} A_n)^c$.

The two inclusions prove the result. □