Homework #15 Solutions
Due Monday, December 1

Exercise 7.2.1. Let f be a bounded function on $[a, b]$, and let P be an arbitrary partition of $[a, b]$. First, explain why $U(f, P) \geq L(f, P)$. Now, prove Lemma 7.2.6.

Solution. *Proof.* Let f be a bounded function on $[a, b]$ and let P be a partition of $[a, b]$. The lower sum of f with respect to P is

$$L(f, P) = \sum_{k=1}^{n} m_k(x_k - x_{k-1})$$

and the upper sum of f with respect to P is

$$U(f, P) = \sum_{k=1}^{n} M_k(x_k - x_{k-1}),$$

where $m_k = \inf\{f(x) : x \in [x_{k-1}, x_k]\}$ and $M_k = \sup\{f(x) : x \in [x_{k-1}, x_k]\}$.

Since $m_k \leq M_k$ for all k, we certainly have

$$L(f, P) \leq U(f, P).$$

By Lemma 7.2.4, for any partitions P_1, P_2 of $[a, b]$,

$$L(f, P_1) \leq U(f, P_2).$$

Taking the supremum of the left side and the infimum of the right side, we get

$$\begin{aligned} L(f) &= \sup\{L(f, P) : P \text{ a partition of } [a, b]\} \\ &\leq \inf\{U(f, P) : P \text{ a partition of } [a, b]\} = U(f). \end{aligned}$$

So, we conclude that the lower integral of f over $[a, b]$ is at most equal to the upper integral of f over $[a, b]$:

$$L(f) \leq U(f).$$

□

Exercise 7.2.2. Consider $f(x) = 1/x$ over the interval $[1, 4]$. Let P be the partition consisting of the points $\{1, 3/2, 2, 4\}$.

(a) Compute $L(f, P)$, $U(f, P)$ and $U(f, P) - L(f, P)$.

(b) What happens to the value of $U(f, P) - L(f, P)$ when we add the point 3 to the partition?

(c) Find a partition P' of $[1, 4]$ for which $U(f, P') - L(f, P') < 2/5$.

Solution. Consider $f(x) = 1/x$ over the interval $[1, 4]$.

(a) Let P be the partition consisting of the points $\{1, 3/2, 2, 4\}$. Then the intervals formed by the partition are

$$[1, 3/2], \quad [3/2, 2], \quad [2, 4].$$

For these intervals, we compute

$$\begin{aligned}
[1, 3/2]: \quad & m_1 = 2/3, \\
& M_1 = 1; \\
[3/2, 2]: \quad & m_2 = 1/2, \\
& M_2 = 2/3; \\
[2, 4]: \quad & m_3 = 1/4, \\
& M_3 = 1/2.
\end{aligned}$$

Now we compute

$$\begin{aligned}
L(f, P) &= m_1(x_1 - x_0) + m_2(x_2 - x_1) + m_3(x_3 - x_3) \\
&= \frac{2}{3} \cdot ((3/2) - 1) + \frac{1}{2} \cdot (2 - (3/2)) + \frac{1}{4} \cdot (4 - 2) \\
&= \frac{13}{12},
\end{aligned}$$

and

$$\begin{aligned}
U(f, P) &= M_1(x_1 - x_0) + M_2(x_2 - x_1) + M_3(x_3 - x_3) \\
&= 1 \cdot ((3/2) - 1) + \frac{2}{3} \cdot (2 - (3/2)) + \frac{1}{2} \cdot (4 - 2) \\
&= \frac{11}{6},
\end{aligned}$$

So, we see the difference $U(f, P) - L(f, P) = \frac{3}{4}$.

(b) If we add the point 3 to the partition, the intervals formed by the partition are

$$[1, 3/2], \quad [3/2, 2], \quad [2, 3], \quad [3, 4].$$

For these intervals, we compute

$$\begin{aligned}
[1, 3/2]: \quad & m_1 = 2/3, \\
& M_1 = 1; \\
[3/2, 2]: \quad & m_2 = 1/2, \\
& M_2 = 2/3; \\
[2, 3]: \quad & m_3 = 1/3, \\
& M_3 = 1/2; \\
[3, 4]: \quad & m_4 = 1/4, \\
& M_4 = 1/3.
\end{aligned}$$

Now we compute

$$\begin{aligned}
L(f, P) &= m_1(x_1 - x_0) + m_2(x_2 - x_1) + m_3(x_3 - x_2) + m_4(x_4 - x_3) \\
&= \frac{2}{3} \cdot ((3/2) - 1) + \frac{1}{2} \cdot (2 - (3/2)) + \frac{1}{3} \cdot (3 - 2) + \frac{1}{4} \cdot (4 - 3) \\
&= \frac{7}{6},
\end{aligned}$$

and

$$\begin{aligned}
U(f, P) &= M_1(x_1 - x_0) + M_2(x_2 - x_1) + M_3(x_3 - x_2) + M_4(x_4 - x_3) \\
&= 1 \cdot ((3/2) - 1) + \frac{2}{3} \cdot (2 - (3/2)) + \frac{1}{2} \cdot (3 - 2) + \frac{1}{3} \cdot (4 - 3) \\
&= \frac{5}{3},
\end{aligned}$$

So, we see the difference $\frac{5}{3} - \frac{7}{6} = \frac{1}{2}$.

(c) All you have to do is add any point to the partition. I'll let you do that.

Exercise 7.2.3 (Sequential Criterion for Integrability)**.** (a) Prove that a bounded function f is integrable on $[a, b]$ if and only if there exists a sequence of partitions $(P_n)_{n=1}^{\infty}$ satisfying

$$\lim_{n\to 0}\,[U(f, P_n) - L(f, P_n)] = 0.$$

(b) For each n, let P_n be the partition of $[0, 1]$ into n equal subintervals. Find formulas for $U(f, P_n)$ and $L(f, P_n)$ if $f(x) = x$. The formula $1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$ will be useful.

(c) Use the sequential criterion for integrability from (a) to show directly that $f(x) = x$ is integrable on $[0, 1]$ and compute $\int_0^1 f$.

Solution. (a) *Proof.* ($\Leftarrow$) Let f be an integrable function on $[a, b]$. Let $n \in \mathbb{N}$. By Theorem 7.2.8, there exists a partition P_n of $[a, b]$ so that

$$0 \le U(f, P_n) - L(f, P_n) < \frac{1}{n}.$$

Taking the limit as $n \to \infty$ using the Squeeze Theorem, we get

$$\lim_{n\to\infty}[U(f, P_n) - L(f, P_n)] = 0.$$

($\Rightarrow$) Let f be a bounded function on $[a, b]$ satisfying the property that there exists a sequence of partitions $(P_n)_{n=1}^{\infty}$ satisfying

$$\lim_{n\to 0}\,[U(f, P_n) - L(f, P_n)] = 0.$$

By definition of upper and lower integrals, we have

$$L(f, P_n) \le L(f) \le U(f) \le U(f, P_n),$$

so that

$$U(f) - L(f) \le U(f, P_n) - L(f, P_n)$$

for all $n \in \mathbb{N}$. Taking the limit as $n \to \infty$ and noting that $U(f) - L(f) \ge 0$, we get

$$0 \le U(f) - L(f, P) \le \lim_{n\to 0}\,[U(f, P_n) - L(f, P_n)] = 0.$$

So, $U(f) = L(f)$ and f is integrable. □

(b) For each k, $1 \leq k \leq n$, let $x_k = k/n$ and let $P_n = \{x_1, x_2, \ldots, x_n\}$. Noting that on the kth interval, $[x_{k-1}, x_k]$, $m_k = x_{k-1}$ and $M_k = x_k$. So, the upper sum is

$$\begin{aligned}
U(f, P_n) &= \sum_{k=1}^{n} M_k(x_k - x_{k-1}) = \sum_{k=1}^{n} x_k(x_k - x_{k-1}) \\
&= \sum_{k=1}^{n} \frac{k}{n}\left(\frac{k}{n} - \frac{k-1}{n}\right) = \sum_{k=1}^{n} \frac{k}{n^2} = \frac{1}{n^2}\sum_{k=1}^{n} k \\
&= \frac{1}{n^2} \cdot \frac{n(n+1)}{2} \\
&= \frac{n+1}{2n}.
\end{aligned}$$

and the lower sum is

$$\begin{aligned}
L(f, P_n) &= \sum_{k=1}^{n} m_k(x_k - x_{k-1}) = \sum_{k=1}^{n} x_{k-1}(x_k - x_{k-1}) \\
&= \sum_{k=1}^{n} \frac{k-1}{n}\left(\frac{k}{n} - \frac{k-1}{n}\right) = \sum_{k=1}^{n} \frac{k-1}{n^2} = \frac{1}{n^2}\sum_{k=1}^{n} (k-1) \\
&= \frac{1}{n^2} \cdot \frac{n(n-1)}{2} \\
&= \frac{n-1}{2n}.
\end{aligned}$$

(c) From part (b), we have

$$U(f, P_n) - L(f, P_n) = \frac{n+1}{2n} - \frac{n-1}{2n} = \frac{1}{n}.$$

Notice we have

$$\lim_{n\to\infty}[U(f, P_n) - L(f, P_n)] = \lim_{n\to\infty} \frac{1}{n} = 0.$$

By part (a), f is integrable.

Exercise 7.2.5. Assume that, for each n, f_n is an integrable function on $[a, b]$. If $(f_n) \to f$ uniformly on $[a, b]$, prove that f is also integrable on this set. (We will see that this conclusion does not necessarily follow if the convergence is pointwise.)

Solution. *Proof.* For each $n \in \mathbb{N}$, suppose f_n is an integrable function on $[a, b]$ and suppose that(f_n) converges to f uniformly on $[a, b]$. Let $\epsilon > 0$.

Since (f_n) converges uniformly to f on $[a, b]$, there exists $N \in \mathbb{N}$ so that whenever $n \geq N$, we have

$$|f_n(x) - f(x)| < \frac{\epsilon}{3(b-a)}$$

for all x in $[a, b]$.

Since f_N is integrable on $[a, b]$, there exists a partition P of $[a, b]$ so that

$$U(f_N, P) - L(f_N, P) < \frac{\epsilon}{3}.$$

By this choice of N, it follows that

$$|M_k(f) - M_k(f_N)| \leq \frac{\epsilon}{3(b-a)}$$
$$|m_k(f) - m_k(f_N)| \leq \frac{\epsilon}{3(b-a)},$$

where $M_k(f) = \sup\{f(x) \mid x \in [x_{k-1}, x_k]\}$, $m_k(f) = \inf\{f(x) \mid x \in [x_{k-1}, x_k]\}$, and similarly for f_N.

Then we have

$$\begin{aligned}
|L(f, P) - L(f_N, P)| &= \left| \sum_{k=1}^{n} (m_k(f) - m_k(f_N))\, \Delta x_k \right| \\
&\leq \sum_{k=1}^{n} |m_k(f) - m_k(f_N)|\, \Delta x_k \\
&\leq \sum_{k=1}^{n} \frac{\epsilon}{3(b-a)}\, \Delta x_k \\
&\leq \frac{\epsilon}{3(b-a)} \sum_{k=1}^{n} \Delta x_k \\
&\leq \frac{\epsilon}{3(b-a)}(b-a) \\
&\leq \frac{\epsilon}{3},
\end{aligned}$$

and

$$
\begin{aligned}
|U(f,P) - U(f_N,P)| &= \left| \sum_{k=1}^{n} (M_k(f) - M_k(f_N))\, \Delta x_k \right| \\
&\le \sum_{k=1}^{n} |M_k(f) - M_k(f_N)|\, \Delta x_k \\
&\le \sum_{k=1}^{n} \frac{\epsilon}{3(b-a)}\, \Delta x_k \\
&\le \frac{\epsilon}{3(b-a)} \sum_{k=1}^{n} \Delta x_k \\
&\le \frac{\epsilon}{3(b-a)} (b-a) \\
&\le \frac{\epsilon}{3}.
\end{aligned}
$$

Finally, we have

$$
\begin{aligned}
|U(f,P) - L(f,P| &= |U(f,P) - U(f_N,P) + U(f_N,P) - L(f_N,P) + L(f_N,P) - L(f,P| \\
&\le |U(f,P) - U(f_N,P)| + |U(f_N,P) - L(f_N,P)| + |L(f_N,P) - L(f,P| \\
&< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\
&< \epsilon.
\end{aligned}
$$

Since $\epsilon > 0$ is arbitrary, f is integrable by Theorem 7.2.8. □

Exercise 7.3.3. Let

$$f(x) = \begin{cases} 1 & \text{if } x = 1/n \text{ for some } n \in \mathbb{N} \\ 0 & \text{otherwise.} \end{cases}$$

Show that f is integrable on $[0, 1]$ and compute $\int_0^1 f$.

Solution. For any partition P, it is clear that

$$L(f, P) = 0$$

since any subinterval contains an irrational number where f is defined to be zero.

Take partition P_n consisting of points given by $x_k = k/n^2$ and $x_0 = 0$. The length of this interval is $\Delta x_k = 1/n^2$ so

$$\begin{aligned} U(f, P_n) &= \frac{1}{n^2}(1 + \cdots + 1) + \frac{1}{n^2} \sup \left\{ f(t) : t \le \frac{1}{n} \right\} \\ &= \frac{n}{n^2} + \frac{1}{n^2} \\ &= \frac{1}{n} + \frac{1}{n^2}. \end{aligned}$$

Given any $\epsilon > 0$, we may find N such that $\frac{1}{N} + \frac{1}{N^2} < \epsilon$ thus for all $n \ge N$, $U(f, P_n) < \epsilon$. This shows

$$U(f, P) = 0,$$

that is

$$\int_0^1 f = U(f, P) = L(f, P) = 0.$$[1]

[1]Proof shamlessly stolen from quizlet.com