Homework #14 Solutions
Due Monday, November 17

Exercise 6.5.3. Use the Weierstrass M-Test to prove Theorem 6.5.2.

Theorem. *If a power series* $\sum_{n=0}^{\infty} a_n x^n$ *converges absolutely at a point* x_0*, then it converges uniformly on the closed interval* $[-c, c]$ *where* $c = |x_0|$*.*

Solution. *Proof.* Suppose a power series $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely at a point x_0. If $x_0 = 0$ there is nothing to prove, so we may suppose $x_0 \neq 0$.

Let $c = |x_0|$. Let $M_n = |a_n| c^n$.

Since $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely at a point x_0, the series

$$\sum_{n=0}^{\infty} |a_n x_0^n| = \sum_{n=0}^{\infty} |a_n| c^n = \sum_{n=0}^{\infty} M_n$$

converges.

Let $x \in [-c, c]$. Then

$$|a_n x^n| = |a_n| \, |x|^n \leq |a_n| c^n = M_n.$$

By the Weierstrass M-Test, the series $\sum_{n=0}^{\infty} a_n x^n$ converges uniformly on $[-c, c]$.

□

Exercise 6.5.5. (a) If s satisfies $0 < s < 1$, show ns^{n-1} is bounded for all $n \geq 1$.

(b) Given an arbitrary $x \in (-R, R)$, pick t to satisfy $|x| < t < R$. Use this start to construct a proof for Theorem 6.5.6.

Theorem. *If* $\sum_{n=0}^{\infty} a_n x^n$ *converges for all* $x \in (-R, R)$*, then the differentiated series* $\sum_{n=1}^{\infty} n a_n x^{n-1}$ *converges at each* $x \in (-R, R)$ *as well. Consequently, the convergence is uniform on compact sets contained in* $(-R, R)$*.*

Solution. (a) *Proof.* Consider the power series $\sum_{n=1}^{\infty} n x^{n-1}$. Applying the Ratio Test, we compute

$$\begin{aligned} \lim_{n\to\infty} \left| \frac{b_{n+1}}{b_n} \right| &= \lim_{n\to\infty} \left| \frac{(n+1)x^n}{nx^{n-1}} \right| \\ &= \lim_{n\to\infty} \left(\frac{n+1}{n} \right) |x| \\ &= |x|. \end{aligned}$$

By the Ratio Test, this power series converges absolutely on the interval $(-1, 1)$.

Let $0 < s < 1$. Then the series $\sum_{n=1}^{\infty} ns^{n-1}$ converges. But this means that the terms are bounded. So, there exists $M \in \mathbb{R}$ so that $ns^{n-1} \leq M$ for all $n \in \mathbb{N}$. □

(b) *Proof.* Let $x \in (-R, R)$, pick t to satisfy $|x| < t < R$ and let $r = |x|/t$, which we note is less than one. We observe that

$$|na_n x^{n-1}| = \frac{1}{t} \left(n \left| \frac{x^{n-1}}{t^{n-1}} \right| \right) |a_n t^n| = \frac{1}{t} \left(n|r|^{n-1} \right) |a_n t^n|.$$

Applying part (a), there exists $M > 0$ so that $n|r|^{n-1} \leq M$ for all $n \in \mathbb{N}$. Then

$$|na_n x^{n-1}| \leq \frac{M}{t} |a_n t^n|.$$

Since t lies in the interval of convergence, the series $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely for $x = t$. So

$$\sum_{n=0}^{\infty} |a_n t^n|$$

converges. But then the multiple $\sum \frac{M}{t} |a_n t^n| = \frac{M}{t} \sum |a_n t^n|$ also coverges.

Since $\sum \frac{M}{t} |a_n t^n|$ converges and $|na_n x^{n-1}| \leq \frac{M}{t} |a_n t^n|$ for all $n \in \mathbb{N}$, we have

$$\sum_{n=1}^{\infty} n a_n x^{n-1}$$

converges absolutely by the Comparison Test. □

Exercise 6.6.2. Starting from one of the previously generated series in this section, use manipulations similar to those in Example 6.6.1 to find a Taylor series representations for each of the following functions. For precisely what values of x is each series representation valid?

(a) $x\cos(x^2)$

(b) $x/(1+4x^2)^2$

(c) $\ln(1+x^2)$

Solution. (a) We have already computed that

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots .$$

Substituting x^2 for x, we have

$$\cos(x^2) = 1 - \frac{x^4}{2!} + \frac{x^8}{4!} - \frac{x^{12}}{6!} + \frac{x^{16}}{8!} - \cdots .$$

Multiplying by x, we get

$$x\cos(x^2) = x - \frac{x^5}{2!} + \frac{x^9}{4!} - \frac{x^{13}}{6!} + \frac{x^{17}}{8!} - \cdots .$$

(b) If we look at the geometric series with ratio $-4x^2$, we get

$$\frac{1}{1+4x^2} = 1 - 4x^2 + 16x^4 - 64x^6 + 256x^8 - \cdots$$

This converges absolutely for $|x| < 1/2$. Taking the derivative, we get

$$\frac{-8x}{(1+4x^2)^2} = -8x + 64x^3 - 384x^5 + 2048x^7 - \cdots$$

Divide by -8 to get

$$\frac{x}{(1+4x^2)^2} = x - 8x^3 + 48x^5 - 256x^7 + \cdots$$

(c) If we look at the geometric series with ratio $-x^2$, we get

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + x^8 - \cdots$$

This converges absolutely for $|x| < 1$. Multiplying by $2x$, we get

$$\frac{2x}{1+x^2} = 2x - 2x^3 + 2x^5 - 2x^7 + 2x^9 - \cdots$$

Taking the antiderivative, we get

$$\ln(1+x^2) = x^2 - \frac{1}{2}x^4 + \frac{1}{3}x^6 - \frac{1}{4}x^8 + \frac{1}{5}x^{10} - \cdots$$

Exercise 6.6.7. Find an example of each of the following or explain why no such function exists.

(a) An infinitely differentiable function $g(x)$ on all of $\mathbb{R}$ with a Taylor series that converges to $g(x)$ only for $x \in (-1, 1)$.

(b) An infinitely differentiable function $h(x)$ with the same Taylor series as $\sin x$ but such that $h(x) \neq \sin x$ for all $x \neq 0$.

(c) An infinitely differentiable function $f(x)$ on all of $\mathbb{R}$ with a Taylor series that converges to $f(x)$ if and only if $x \leq 0$.

Solution. (a) Let $g(x) = \frac{1}{1+x^2}$. Then

$$g(x) = 1 - x^2 + x^4 - x^6 + x^8 - \cdots .$$

for $|x| < 1$.

(b) Let $g(x)$ be the counterexample introduced on p. 203.

$$g(x) = \begin{cases} e^{-1/x^2} & \text{for } x \neq 0, \\ 0 & \text{for } x = 0. \end{cases}$$

The Maclaurin series for g is identically zero, but g is nowhere zero except at $x = 0$.

Let $h(x) = \sin x + g(x)$. Then h and $\sin x$ have the same Maclaurin series, but the two functions are not equal except at $x = 0$.

(c) Modify the counterexample introduced on p. 203:

$$f(x) = \begin{cases} e^{-1/x^2} & \text{for } x > 0, \\ 0 & \text{for } x \leq 0. \end{cases}$$

The Maclaurin series for f is identically zero, but f is nowhere zero except $x < 0$, where it equals its Taylor series (which is identically 0).

Exercise 6.6.10. Consider $f(x) = 1/\sqrt{1-x}$.

(a) Generate the Taylor series for f centered at zero, and use Lagrange's Remainder Theorem to show the series converges to f on $[0, 1/2]$. (The case $x < 1/2$ is more straightforward while $x = 1/2$ requires some extra care.). What happens when we attempt this with $x > 1/2$?

(b) Use Cauchy's Remainder Theorem proved in Exercise 6.6.9 to show the series representation for f holds on $[0, 1)$. (You do not have to do Exercise 6.6.9. Just use it here.)

Solution. (a) We have

$$f^{(n)}(x) = \frac{\prod_{i=1}^{n}(2i-1)}{2^n}\left(\frac{1}{1-x}\right)^n\sqrt{\frac{1}{1-x}}.$$

The Maclaurin series is

$$\sum_{n=0}^{\infty}\frac{\prod_{i=1}^{n}(2i-1)}{2^n n!}x^n = \sum_{n=0}^{\infty}\left(\frac{x^n}{2^n}\prod_{i=1}^{n}\frac{(2i-1)}{i}\right) \le \sum_{n=0}^{\infty}\left(\frac{x^n}{2^n}\prod_{i=1}^{n}2\right) = \sum_{n=0}^{\infty}x^n,$$

so we know that the Maclaurin series at least converges to something for $x \in [0, 1)$.

Lagrange's form of the remainder gives us

$$E_N(x) = \frac{\left(\prod_{i=1}^{N+1}(2i-1)\right)}{2^{N+1}(N+1)!}\left(\frac{1}{1-c}\right)^{N+1}\sqrt{\frac{1}{1-c}}\,x^{N+1}$$

for some $c \in (0, x)$. For $x = 1/2$ and $0 < c < x$:

$$\begin{aligned}|E_N(x)| &\le \left(\prod_{i=1}^{N+1}\frac{2i-1}{i}\right)\left(\frac{1}{2-2c}\right)^{N+1}\left(\sqrt{\frac{1}{1-c}}\right)\frac{1}{2^{N+1}}\\ &\le 2^{N+1}d^{N+1}\left(\sqrt{\frac{1}{1-c}}\right)\frac{1}{2^{N+1}} = d^{N+1}\sqrt{\frac{1}{1-c}},\end{aligned}$$

where $d = 1/(2-2c) < 1$; this shows E_N converges to 0 over $[0, 1/2]$.

Writing E_N in product notation,

$$E_N(x) = \prod_{i=1}^{N+1}\frac{(2i-1)x}{2i(1-c)}.$$

If $x > 1/2$, then it's possible for $\frac{x}{1-c} > 1$. Then for some I, for $i > I$ we have

$$\frac{x}{1-c} > \frac{2i}{2i-1} > 1.$$

Beyond that point, the terms in the product begin increasing, with the product as a whole growing exponentially and diverging.

(b) Plugging in Cauchy's Remainder Theorem,

$$\begin{aligned} E_N(x) &= \left(\prod_{i=1}^{N+1} \frac{2i-1}{i}\right) \frac{(2N+1)(x-c)^N x}{2^N (1-c)^{N+1}} \sqrt{\frac{1}{1-c}} \\ &\leq (2N+1)(d)^N \frac{x}{1-c} \sqrt{\frac{1}{1-c}}, \end{aligned}$$

where $d = \frac{x-c}{1-c} < 1$. The first term is linear in N, the second is exponentially decaying in N, and the last two terms are constant, so the behavior is dominated by the exponential decay and $E_N(x)$ converges to 0.[1]

[1] Shamelessly stolen from https://www.uli.rocks/understanding-analysis-solutions/main.pdf and edited.