Homework #13
Due Monday, November 10

Exercise 6.3.1. Consider the sequence of functions defined by

$$g_n(x) = \frac{x^n}{n}.$$

(a) Show that (g_n) converges uniformly on $[0,1]$ and find $g = \lim g_n$. Show that g is differentiable and compute $g'(x)$ for all $x \in [0,1]$.

(b) Now, show that (g_n') converges on $[0,1]$. Is the convergence uniform? Set $h = \lim g_n'$ and compare h and g'. Are they the same?

Solution. Let

$$g_n(x) = \frac{x^n}{n}.$$

(a) *Proof.* For $x \in [0,1]$, $g_n(x)$ converges pointwise to $g \equiv 0$. Let's show this convergence is uniform.

$$|g_n(x) - g(x)| = \frac{x^n}{n} \leq \frac{1}{n}$$

for all $x \in [0,1]$. So, we see that (g_n) converges to g uniformly on $[0,1]$. □

(b) *Proof.* We compute

$$g_n'(x) = x^{n-1}.$$

For $0 \leq x < 1$, $g_n'(x) \to 0$. However, $g_n'(1) = 1$ for all $n \in \mathbb{N}$. So, (g_n') converges to the function

$$h(x) = \begin{cases} 0 & \text{for } 0 \leq x < 1 \\ 1 & \text{for } x = 1. \end{cases}$$

Since g_n' is continuous for all $n \in \mathbb{N}$ and h is not, this convergence cannot be uniform, by Theorem 6.2.6. Since the limit function g is identically zero, g' is also identically zero. So we see that the sequence (g_n') does not converge to g'. □

Exercise 6.3.3. Consider the sequence of functions

$$f_n(x) = \frac{x}{1+nx^2}.$$

(a) Find the points on $\mathbb{R}$ where each $f_n(x)$ attains its maximum and minimum value. Use this to prove (f_n) converges uniformly on $\mathbb{R}$. What is the limit function?

(b) Let $f = \lim f_n$. Compute $f_n'(x)$ and find all the values of x for which $f'(x) = \lim f_n'(x)$.

Solution. We claim that (f_n) converges uniformly on $\mathbb{R}$ to $f \equiv 0$. Let's show this.

Using some calculus and graphing the function, we see that $|f_n(x)| \leq \frac{1}{2\sqrt{n}}$.

Let $\epsilon > 0$ and choose $N > \frac{1}{4\epsilon^2}$. Then for $n \geq N$, we have

$$|f_n(x)| \leq \frac{1}{2\sqrt{n}} \leq \frac{1}{2\sqrt{N}} < \epsilon$$

for all x. This shows the sequence (f_n) converges to zero uniformly on the entire real line.

We compute that

$$f_n'(x) = \frac{1-nx^2}{(1+nx^2)^2}.$$

Notice the derivative is defined for all $x \in \mathbb{R}$ since $1 + nx^2 \neq 0$ for all $x \in \mathbb{R}$. Taking the limit as $n \to \infty$, we get

$$\begin{aligned}\lim_{n\to\infty} \frac{1-nx^2}{(1+nx^2)^2} &= \lim_{n\to\infty} \frac{1-nx^2}{(1+nx^2)^2} \cdot \frac{1/n^2}{1/n^2} \\ &= \lim_{n\to\infty} \frac{\frac{1}{n^2} - \frac{x^2}{n}}{(\frac{1}{n} + x^2)^2} \\ &= 0,\end{aligned}$$

provided $x \neq 0$. If $x = 0$, we see this limit is 1.

So, we see that $(f_n'(x))$ converges to $f'(x)$ for $x \neq 0$, but not at $x = 0$.

Exercise 6.4.2. Decide whether each proposition is true or false, providing a short justification or counterexample as appropriate.

(a) If $\sum_{n=1}^{\infty} g_n$ converges uniformly, then (g_n) converges uniformly to zero.

(b) If $0 \leq f_n(x) \leq g_n(x)$ and $\sum_{n=1}^{\infty} g_n$ converges uniformly, then $\sum_{n=1}^{\infty} f_n$ converges uniformly.

(c) If $\sum_{n=1}^{\infty} f_n$ converges uniformly on A, then there exist constants M_n such that $|f_n(x)| \leq M_n$ for all $x \in A$ and $\sum_{n=1}^{\infty} M_n$ converges.

Solution. (a) *Proof.* Let $\sum_{n=1}^{\infty} g_n$ converge uniformly on some set A.

Let $\epsilon > 0$. By the Cauchy Criterion for Uniform Convergence of Series (Theorem 6.4.4), there exists $N \in \mathbb{N}$ so that for all $n > m \geq N - 1$,

$$|g_{m+1}(x) + g_{m+2}(x) + \cdots + g_n(x)| < \epsilon \quad \text{for all } x \in A. \tag{1}$$

Let $n \geq N$ and let $m = n - 1$. Note that $n > m \geq N - 1$, so that by Equation (1),

$$|g_n(x)| < \epsilon \quad \text{for all } x \in A.$$

Since $n \geq N$ is arbitrary, this later inequality holds for all $n \geq N$, and since $\epsilon > 0$ is arbitrary, (g_n) converges to zero uniformly on A. □

(b) *Proof.* Suppose $0 \leq f_n(x) \leq g_n(x)$ for all $n \in \mathbb{N}$ and $\sum_{n=1}^{\infty} g_n$ converges uniformly on a set A.

Let $\epsilon > 0$. By the Cauchy Criterion for Uniform Convergence of Series (Theorem 6.4.4) there exists $N \in \mathbb{N}$ so that

$$|g_{m+1}(x) + g_{m+2}(x) + \cdots + g_n(x)| < \epsilon.$$

whenever $n > m \geq N$ and for all $x \in A$.

Let $n > m \geq N$. Then

$$\begin{aligned} |f_{m+1}(x) + f_{m+2}(x) + \cdots + f_n(x)| &= f_{m+1}(x) + f_{m+2}(x) + \cdots + f_n(x) \\ &\leq g_{m+1}(x) + g_{m+2}(x) + \cdots + g_n(x) \\ &\leq |g_{m+1}(x) + g_{m+2}(x) + \cdots + g_n(x)| \\ &< \epsilon \end{aligned}$$

for all $x \in A$. Since $\epsilon > 0$ is arbitrary, the series $\sum_{n=1}^{\infty} f_n$ converges uniformly on A by the Cauchy Criterion for Uniform Convergence of Series (Theorem 6.4.4). □

(c) This statement is false.

As a counterexample, for each $n \in \mathbb{N}$, let

$$f_n : \mathbb{R} \to \mathbb{R}$$
$$f_n(x) = \begin{cases} \dfrac{1}{n} & \text{if } n-1 < x \leq n \\ 0 & \text{otherwise} \end{cases}$$

We first show that $\sum_{n=1}^{\infty} f_n$ converges uniformly on $\mathbb{R}$. We remark that for a fixed value of x, $f_n(x) \neq 0$ for only one value of n, so that

$$|f_{m+1}(x) + \cdots + f_n(x)| < \frac{1}{m}$$

for all $x \in \mathbb{R}$.

Let $\epsilon > 0$. Choose $N > 1/\epsilon$. Then for $n > m \geq N$, we have

$$|f_{m+1}(x) + \cdots + f_n(x)| < \frac{1}{m} \leq \frac{1}{N} < \epsilon.$$

The sum $\sum_{n=1}^{\infty} f_n$ converges uniformly by the Cauchy Criterion for Uniform Convergence of Series.

However, if $|f_n(x)| \leq M_n$ for all x, then $M_n \geq 1/n$, and so $\sum_{n=1}^{\infty} M_n$ diverges by the Comparison Test.[1]

[1] This example is freely stolen and adapted from Math Stack Exchange under the topic "Converse of the Weierstrass M-Test."

Exercise 6.4.7. Let

$$f(x) = \sum_{k=1}^{\infty} \frac{\sin(kx)}{k^3}.$$

(a) Show that $f(x)$ is differentiable and that the derivative $f'(x)$ is continuous.

(b) Can we determine if f is twice-differentiable?

Solution. Let

$$f(x) = \sum_{k=1}^{\infty} \frac{\sin(kx)}{k^3}$$

(a) Let $f_k(x) = \sin(kx)/k^3$. Then $|f_k(x)| \leq 1/k^3$ for all $x \in \mathbb{R}$ and $\sum_{k=1}^{\infty} 1/k^3$ converges since it's a p-series with $p = 3 > 1$. So by the Weierstrass M-Test, the series $\sum_{k=1}^{\infty} \sin(kx)/k^3$ converges uniformly on the entire real line.

Now, we compute $f_k'(x) = \cos(kx)/k^2$. Then $|f_k'(x)| \leq 1/k^2$ for all $x \in \mathbb{R}$ and $\sum_{k=1}^{\infty} 1/k^2$ converges since it's a p-series with $p = 2 > 1$. So by the Weierstrass M-Test, the series $\sum_{k=1}^{\infty} \cos(kx)/k^2$ converges uniformly on the entire real line.

Since each f_k' is continuous and $g = \sum f_k'$ converges uniformly, g is continuous by Theorem 6.4.2. By Theorem 6.4.3, $f' = g$. So, f is continuously differentiable.

(b) If we take the second derivative of f_k, we get

$$f_k''(x) = -\frac{\sin(kx)}{k}$$

and we see that $|f''(x)| \leq 1/k$ for all x. However, the series $\sum_{k=1}^{\infty} 1/k$ does not converge, so the Weierstrass M-Test can't be applied. Thus, we cannot use Theorems 6.4.2 and 6.4.3 as we did before. So, my guess is no.

Exercise 6.4.9. Let
$$h(x) = \sum_{n=1}^{\infty} \frac{1}{x^2 + n^2}.$$

(a) Show that h is a continuous function defined on all of $\mathbb{R}$.

(b) Is h differentiable? If so, is the derivative function h' continuous?

Solution. (a) Let
$$h(x) = \sum_{n=1}^{\infty} \frac{1}{x^2 + n^2}.$$

Let
$$h_n(x) = \frac{1}{x^2 + n^2}.$$

We note that $h_n(x)$ is defined and continuous for all $x \in \mathbb{R}$. We also note that
$$|h_n(x)| = \left| \frac{1}{x^2 + n^2} \right| \leq \frac{1}{n^2}$$
for all $n \in \mathbb{N}$. Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges (since it's a p-series with $p = 2 > 1$), the series
$$h(x) = \sum_{n=1}^{\infty} \frac{1}{x^2 + n^2}.$$
converges uniformly for all $x \in \mathbb{R}$ by the Weierstrass M-Test. By Theorem 6.4.2, h is continuous on $\mathbb{R}$.

(b) Using the notation and definitions in part (a), we compute
$$h_n'(x) = -\frac{2x}{(x^2 + n^2)^2}.$$

Let $R > 0$ be arbitrary and consider h_n' on the interval $[-R, R]$. On this interval
$$|h_n'(x)| = \left| \frac{2x}{(x^2 + n^2)^2} \right| = \frac{2|x|}{(x^2 + n^2)^2} \leq \frac{2R}{n^4}.$$

Since the series $\sum_{n=1}^{\infty} 2R/n^4 = 2R \sum_{n=1}^{\infty} 1/n^4$ converges (since it's a multiple of a p-series with $p = 4 > 1$), the series $\sum_{n=1}^{\infty} h_n'$ converges uniformly on $[-R, R]$ to some function g. By Theorem 6.4.3, h is differentiable on $[-R, R]$ and
$$h'(x) = \sum_{n=1}^{\infty} \frac{-2x}{(x^2 + n^2)^2}.$$

Since $R > 0$ is arbitrary, h is differentiable on $\mathbb{R}$ and $h'(x) = \sum_{n=1}^{\infty} -2x/(x^2 + n^2)^2$. Since each h_n' is continuous and the convergence of $\sum_{n=1}^{\infty} h_n'$ is uniform on any interval $[-R, R]$, we have h' is also continuous on $\mathbb{R}$.