Homework #12
Due Monday, November 3

Exercise 5.3.11. (a) Use the Generalized Mean Value theorem to furnish a proof of the 0/0 case of L'Hospital's rule (Theorem 5.3.6).

(b) If we keep the first part of the hypothesis of Theorem 5.3.6 the same but we assume that

$$\lim_{x\to a} \frac{f'(x)}{g'(x)} = \infty$$

does it necessarily follow that

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \infty?$$

Solution.

Theorem (L'Hospital's Rule: 0/0 case)**.** *Assume f and g are continuous functions defined on an interval containing a, and assume that f and g are differentiable on this interval, with the possible exception of the point a. If $f(a) = 0$ and $g(a) = 0$, then*

$$\lim_{x\to a} \frac{f'(x)}{g'(x)} = L \quad \textit{implies} \quad \lim_{x\to a} \frac{f(x)}{g(x)} = L.$$

(a) *Proof.* Let f and g be continuous functions defined on an interval containing a, and assume that f and g are differentiable on this interval, with the possible exception of the point a. Suppose further that $f(a) = 0$ and $g(a) = 0$ and

$$\lim_{x\to a} \frac{f'(x)}{g'(x)} = L.$$

Since $\lim_{x\to a} \frac{f'(x)}{g'(x)} = L$, there is some interval containing a on which $g'(x)$ is never zero. By the Generalized Mean Value Theorem, for x inside this interval, we have that there exists a point c between x and a so that

$$\frac{f'(c)}{g'(c)} = \frac{f(x) - f(a)}{g(x) - g(a)} = \frac{f(x)}{g(x)}.$$

Now, taking the limit as x goes to a and noting that c is between x and a, we get

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(c)}{g'(c)} = \lim_{x\to a}\frac{f'(x)}{g'(x)} = L.$$

This concludes the proof. □

(b) *Proof.* Let f and g be continuous functions defined on an interval containing a, and assume that f and g are differentiable on this interval, with the possible exception of the point a. Suppose further that $f(a) = 0$ and $g(a) = 0$ and

$$\lim_{x\to a}\frac{f'(x)}{g'(x)} = \infty.$$

Let $M > 0$. Since $\lim_{x\to a}\frac{f'(x)}{g'(x)} = \infty$, we can choose $\delta > 0$ so that $\frac{f'(x)}{g'(x)} > M$ whenever $0 < |x - a| < \delta$.

Let $x \in (a, a+\delta)$. Applying the Mean Value Theorem to f on the interval $[a, x]$, we get

$$\frac{f(x)}{g(x)} = \frac{f(x)-f(a)}{g(x)-g(a)} = \frac{f'(c)}{g'(c)}$$

for some $c \in (a, x)$. However, in this interval, we have $\frac{f'(c)}{g'(c)} > M$, so that

$$\frac{f(x)}{g(x)} = \frac{f'(c)}{g'(c)} > M.$$

Since this is true for all $x \in (a, a+\delta)$, we see that

$$\lim_{x\to a^+}\frac{f(x)}{g(x)} = \infty.$$

An analogous proof shows that

$$\lim_{x\to a^-}\frac{f(x)}{g(x)} = \infty.$$

So,

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \infty.$$

□

Exercise 6.2.1. Let
$$f_n(x) = \frac{nx}{1+nx^2}.$$

(a) Find the pointwise limit of (f_n) for all $x \in (0,\infty)$.

(b) Is the convergence uniform on $(0,\infty)$?

(c) Is the convergence uniform on $(0,1)$?

(d) Is the convergence uniform on $(1,\infty)$?

Solution. (a) Let $x \in (0,\infty)$. We modify f_n by dividing the numerator and denominator by n.
$$\begin{aligned} f_n(x) &= \frac{nx}{1+nx^2} \\ &= \frac{nx}{1+nx^2} \cdot \frac{\frac{1}{n}}{\frac{1}{n}} \\ &= \frac{x}{\frac{1}{n}+x^2}. \end{aligned}$$

Now, as $n \to \infty$, we see that
$$\frac{x}{\frac{1}{n}+x^2} \to \frac{x}{x^2} = \frac{1}{x}.$$

(b) We compute
$$\begin{aligned} |f_n(x) - f(x)| &= \left| \frac{nx}{1+nx^2} - \frac{1}{x} \right| \\ &= \frac{1}{x(1+nx^2)}. \end{aligned}$$

Let $x_n = 1/n$. Then
$$\begin{aligned} |f_n(x_n) - f(x_n)| &= \frac{1}{\left(\frac{1}{n}\right)\left(1+n\left(\frac{1}{n}\right)^2\right)} \\ &= \frac{1}{\left(\frac{1}{n}\right)\left(1+\frac{1}{n}\right)} \\ &= \frac{n^2}{n+1} \\ &\geq \frac{1}{2} \end{aligned}$$

for all $n \in \mathbb{N}$.

Thus, for all $n \in \mathbb{N}$, we have found $x_n \in (0,\infty)$ so that $|f_n(x_n) - f(x_n)| \geq \frac{1}{2}$. Hence, (f_n) does not converge uniformly on $(0,\infty)$.

(c) The argument used in (b) (starting at $n = 2$) shows that (f_n) does not converge uniformly on $(0, 1)$.

(d) Using the computation in (b), for $x \geq 1$ we have

$$|f_n(x) - f(x)| = \frac{1}{x(1 + nx^2)} \leq \frac{1}{1 + n}.$$

So, we see the convergence is uniform on the interval $(1, \infty)$.

Exercise 6.2.3. For each $n \in \mathbb{N}$ and $x \in [0, \infty)$, let

$$g_n(x) = \frac{x}{1+x^n} \qquad \text{and} \qquad h_n(x) = \begin{cases} 1 & \text{if } x \geq \dfrac{1}{n} \\ nx & \text{if } 0 \leq x < \dfrac{1}{n}. \end{cases}$$

Answer the following questions for the sequences (g_n) and (h_n):

(a) Find the pointwise limit on $[0, \infty)$.

(b) Explain how we know that the convergence *cannot* be uniform on $[0, \infty)$.

(c) Choose a smaller set over which the convergence is uniform and supply an argument to show that this is indeed the case.

Solution. (a) For (g_n), the pointwise limit on $[0, \infty)$ is

$$g(x) = \begin{cases} x & \text{for } 0 \leq x < 1 \\ \dfrac{1}{2} & \text{for } x = 1 \\ 0 & \text{for } x > 1 \end{cases}$$

For (h_n), the pointwise limit on $[0, \infty)$ is

$$h(x) = \begin{cases} 0 & \text{for } x = 0 \\ 1 & \text{for } x > 0 \end{cases}$$

(b) The convergence cannot be uniform because all the functions g_n and h_n are continuous and the limit of a sequence of continuous functions converging uniformly is continuous. However, h and g are not continuous.

(c) Over $[1, \infty)$ we have $h_n(x) = h(x) = 1$ for all n, thus $|h_n(x) - h(x)| = 0$ for all $x \in [1, \infty)$ so h_n converges uniformly.

Now for g_n. Let $t \in [0, 1)$. Let $\epsilon > 0$.

Since $t \in [0, 1)$, we know from early in the course that (t^n) converges to 0. Choose $N \in \mathbb{N}$ so that $t^n < \epsilon$ for all $n \geq N$.

Let $n \geq N$. Then

$$\left| \frac{x}{1+x^n} - 1 \right| = \left| \frac{x - x(1+x^n)}{1+x^n} \right| = \left| \frac{x^{n+1}}{1+x^n} \right| < |t^{n+1}| < \epsilon$$

for all $x \in [0, t)$.

Exercise 6.2.5. Using the Cauchy Criterion for convergent sequences of real numbers (Theorem 2.6.4), supply a proof for Theorem 6.2.5. (First, define a candidate for $f(x)$, and then argue that $f_n \to f$ uniformly.)

Theorem (Cauchy Criterion for Uniform Convergence)**.** *A sequence of functions* (f_n) *defined on a set* $A \subseteq \mathbb{R}$ *converges uniformly on* A *if and only if for every* $\epsilon > 0$ *there exists an* $N \in \mathbb{N}$ *such that* $|f_n(x) - f_m(x)| < \epsilon$ *for all* m*,* $n \geq N$ *and all* $x \in A$*.*

Solution. *Proof.* ($\Rightarrow$) Suppose a sequence of functions (f_n) defined on a set $A \subseteq \mathbb{R}$ converges uniformly to f on A. By definition of uniform convergence, for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ so that for all $x \in A$, $|f_n(x) - f(x)| < \epsilon/2$ whenever $n \geq N$. Then for m, $n \geq N$, we have

$$\begin{aligned} |f_n(x) - f_m(x)| &= |f_n(x) - f(x) + f(x) - f_m(x)| \\ &\leq |f_n(x) - f(x)| + |f(x) - f_m(x)| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

for all $x \in A$.

($\Leftarrow$) Let (f_n) be a sequence of functions defined on a set $A \subseteq \mathbb{R}$ which has the property that for every $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that $|f_n(x) - f_m(x)| < \epsilon$ for all m, $n \geq N$ and all $x \in A$.

For a fixed $x \in A$, the condition says that the sequence $(f_n(x))$ is a Cauchy sequence in $\mathbb{R}$. By the Cauchy Criterion (Theorem 2.6.4), $(f_n(x))$ converges to some real number. Call this number $f(x)$. This defines a function $f : A \to \mathbb{R}$.

Let $\epsilon > 0$. By the condition in the hypothesis, there exists $N \in \mathbb{N}$ such that $|f_n(x) - f_m(x)| < \epsilon/2$ for all m, $n \geq N$ and all $x \in A$. Letting $m \to \infty$, we get

$$|f_n(x) - f(x)| \leq \frac{\epsilon}{2} < \epsilon,$$

for all $x \in A$. That is, (f_n) converges uniformly to f on A. □

Exercise 6.2.14. A sequence of functions (f_n) defined on a set $E \subseteq \mathbb{R}$ is called **equicontinuous** if for every $\epsilon > 0$ there exists a $\delta > 0$ such that $|f_n(x) - f_n(y)| < \epsilon$ for all $n \in \mathbb{N}$ and $|x - y| < \delta$ in E.

(a) What is the difference between saying that a sequence of functions (f_n) is equicontinuous and just asserting that each f_n in the sequence is individually uniformly continuous?

(b) Give a qualitative explanation for why the sequence $g_n(x) = x^n$ is not equicontinuous on $[0, 1]$. Is each g_n uniformly continuous on $[0, 1]$?

Solution. (a) For equicontinuous functions the same δ works for every function in the sequence, as opposed to individually being uniformly continuous where δ depends on n.

(b) Not equicontinuous since as n increases we need δ to be smaller, hence δ cannot be written independent of n. Each g_n is uniformly continuous however (since g_n is continuous on the compact set $[0, 1]$).

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