Homework #11
Due Monday, October 27

Exercise 5.2.1. Supply proofs for parts (i) and (ii) of Theorem 5.2.4.

Theorem. *Let f and g be functions defined on an interval A, and assume both are differentiable at some point $c \in A$. Then,*

(i) $(f+g)'(c) = f'(c) + g'(c)$,

(ii) $(kf)'(c) = kf'(c)$ for all $k \in \mathbb{R}$,

(iii) $(fg)'(c) = f'(c)g(c) + f(c)g'(c)$

(iv) $(f/g)'(c) = \dfrac{g(c)f'(c) - f(c)g'(c)}{[g(c)]^2}$, provided that $g(c) \neq 0$.

Solution. Let f and g be functions defined on an interval A, and assume both are differentiable at some point $c \in A$.

(i) *Proof.* Computing, we have

$$\begin{aligned}
(f+g)'(c) &= \lim_{x\to c} \frac{(f+g)(x) - (f+g)(c)}{x-c} \\
&= \lim_{x\to c} \frac{f(x) + g(x) - (f(c) + g(c))}{x-c} \\
&= \lim_{x\to c} \frac{f(x) - f(c) + g(x) - g(c)}{x-c} \\
&= \lim_{x\to c} \left[\frac{f(x) - f(c)}{x-c} + \frac{g(x) - g(c)}{x-c}\right] \\
&= \lim_{x\to c} \frac{f(x) - f(c)}{x-c} + \lim_{x\to c} \frac{g(x) - g(c)}{x-c} \\
&= f'(c) + g'(c).
\end{aligned}$$

□

(ii) *Proof.* Computing, we have

$$\begin{aligned}
(kf)'(c) &= \lim_{x\to c} \frac{(kf)(x) - (kf)(c)}{x - c} \\
&= \lim_{x\to c} \frac{kf(x) - kf(c)}{x - c} \\
&= \lim_{x\to c} \frac{k(f(x) - f(c))}{x - c} \\
&= \lim_{x\to c} k\frac{f(x) - f(c)}{x - c} \\
&= k\lim_{x\to c} \frac{f(x) - f(c)}{x - c} = kf'(c).
\end{aligned}$$

□

Exercise 5.2.3. (a) Use Definition 5.2.1 to produce the proper formula for the derivative of $h(x) = 1/x$.

(b) Combine the result in part (a) with the Chain Rule (Theorem 5.2.5) to supply a proof for part (iv) of Theorem 5.2.4.

(c) Supply a direct proof of Theorem 5.2.4 (iv) by algebraically manipulating the difference quotient for (f/g) in a style similar to the proof of Theorem 5.2.4 (iii).

Solution. (a) *Proof.* Let $f(x) = 1/x$. We compute

$$\begin{aligned} f'(c) &= \lim_{x\to c} \frac{f(x)-f(c)}{x-c} \\ &= \lim_{x\to c} \frac{\frac{1}{x}-\frac{1}{c}}{x-c} \\ &= \lim_{x\to c} \frac{c-x}{xc(x-c)} \\ &= \lim_{x\to c} \frac{-(x-c)}{xc(x-c)} \\ &= \lim_{x\to c} \frac{-1}{xc} \\ &= -\frac{1}{c^2}, \end{aligned}$$

provided $c \neq 0$. □

(b) *Proof.* We use part (a), the Product Rule, and the Chain Rule to prove the Quotient Rule.

Let $F(x) = f(x)/g(x) = f(x)[g(x)]^{-1}$. Then

$$\begin{aligned} F'(x) &= f'(x)[g(x)]^{-1} + f(x)\cdot \frac{d}{dx}[g(x)]^{-1} \\ &= f'(x)[g(x)]^{-1} + f(x)\cdot(-1/[g(x)]^2)\cdot g'(x) \\ &= \frac{f'(x)}{g(x)} - \frac{f(x)g'(x)}{[g(x)]^2} \\ &= \frac{f'(x)g(x)-f(x)g'(x)}{[g(x)]^2}. \end{aligned}$$

□

(c) *Proof.* Now, we prove the Quotient Rule directly from the definition of the derivative. Let $Q(x) = f(x)/g(x)$. Then we compute

$$
\begin{aligned}
Q'(x) &= \lim_{x\to c} \frac{Q(x) - Q(c)}{x - c} \\
&= \lim_{x\to c} \frac{\frac{f(x)}{g(x)} - \frac{f(c)}{g(c)}}{x - c} \\
&= \lim_{x\to c} \frac{f(x)g(c) - f(c)g(x)}{g(x)g(c)(x - c)} \\
&= \lim_{x\to c} \frac{f(x)g(c) - f(c)g(c) + f(c)g(c) - f(c)g(x)}{g(x)g(c)(x - c)} \\
&= \lim_{x\to c} \frac{[f(x) - f(c)]g(c) - f(c)[g(x) - g(c)]}{g(x)g(c)(x - c)} \\
&= \lim_{x\to c} \frac{\frac{f(x)-f(c)}{x-c}g(c) - f(c)\frac{g(x)-g(c)}{x-c}}{g(x)g(c)} \\
&= \frac{\lim_{x\to c}\left[\frac{f(x)-f(c)}{x-c}\right]g(c) - f(c)\lim_{x\to c}\left[\frac{g(x)-g(c)}{x-c}\right]}{\lim_{x\to c} g(x)g(c)} \\
&= \frac{f'(c)g(c) - f(c)g'(c)}{[g(c)]^2}.
\end{aligned}
$$

We remark that since g is differentiable at c, it is also continuous at c, so $\lim_{x\to c} g(x) = g(c)$. □

Exercise 5.2.11. Assume that g is differentiable on $[a, b]$ and satisfies $g'(a) < 0 < g'(b)$.

(a) Show that there exists a point $x \in (a, b)$ where $g(a) > g(x)$, and a point $y \in (a, b)$ where $g(y) < g(b)$.

(b) Now complete the proof of Darboux's Theorem started earlier.

Solution. (a) *Proof.* Let g be differentiable on $[a, b]$ and satisfy $g'(a) < 0 < g'(b)$.

Since $g'(a) < 0$, there exists $\delta > 0$ so that

$$\frac{g(x) - g(a)}{x - a} < 0$$

for all $0 < |x - a| < \delta$. Then for $x = a + \delta/2 \in (a, b)$,

$$\frac{g(x) - g(a)}{\delta/2} < 0,$$

which forces $g(x) < g(a)$.

Since $g'(b) > 0$, there exists $\delta > 0$ so that

$$\frac{g(x) - g(b)}{x - b} > 0$$

for all $0 < |x - b| < \delta$. Then for $x = b - \delta/2 \in (a, b)$,

$$\frac{g(x) - g(b)}{-\delta/2} > 0,$$

which forces $g(x) < g(b)$. □

(b)

Theorem (Darboux's Theorem)**.** *If f is differentiable on an interval $[a, b]$, and if α satisfies $f'(a) < \alpha < f'(b)$ (or $f'(a) > \alpha > f'(b)$), there there exists a point $c \in (a, b)$ where $f'(c) = \alpha$.*

Proof. Beginning with the proof in the book ...

We first simplify matters by defining a new function $g(x) = f(x) - \alpha x$ on $[a, b]$. Notice that g is differentiable on $[a, b]$ with $g'(x) = f'(x) - \alpha$. In terms of g, our hypothesis states that $g'(a) < 0 < g'(b)$, and we hope to show that $g'(c) = 0$ for some $c \in (a, b)$.

Since g is differentiable on $[a, b]$, it must be continuous on $[a, b]$. By the Extreme Value Theorem, g has both a maximum and minimum value of $[a, b]$. By part (a), there exists $x \in (a, b)$ with $g(x) < g(a)$ and $y \in (a, b)$ with $g(y) < g(b)$. Hence, the minimum of g on $[a, b]$ doesn't occur at the endpoints of $[a, b]$. Since g is differentiable on (a, b), the minimum must occur at some point $c \in (a, b)$ with $g'(c) = 0$ by the Interior Extremum Theorem. But then $f'(c) = \alpha$, as desired. □

Exercise 5.3.1. Recall from Homework Exercise 4.4.9 that a function $f : A \to \mathbb{R}$ is Lipschitz on A if there exists an $M > 0$ such that

$$\left| \frac{f(x) - f(y)}{x - y} \right| \leq M$$

for all $x \neq y \in A$.

(a) Show that if f is differentiable on a closed interval $[a, b]$ and if f' is continuous on $[a, b]$, then f is Lipschitz on $[a, b]$.

(b) Review the definition of a contractive function in Exercise 4.3.11. If we add the assumption that $|f'(x)| < 1$ on $[a, b]$, does it follow that f is contractive on this set?

Solution. (a) *Proof.* Let f be differentiable on a closed interval $[a, b]$ with f' is continuous on $[a, b]$. Since f' is continuous on the closed interval $[a, b]$, f' has a minimum value m_1 and a maximum value m_2 on this interval by the Extreme Value Theorem.

Let $a \leq x < y \leq b$. Applying the Mean Value Theorem on the interval $[x, y]$, we get

$$m_1 \leq \frac{f(x) - f(y)}{x - y} = f'(\xi) \leq m_2.$$

for some ξ between x and y. Since x and y are arbitrary in $[a, b]$,

$$m_1 \leq \frac{f(x) - f(y)}{x - y} \leq m_2.$$

for all x, y in $[a, b]$. If we take $M = \max\{|m_1|, |m_2|\}$, then

$$\left| \frac{f(x) - f(y)}{x - y} \right| \leq M.$$

Thus, f is Lipschitz on $[a, b]$. □

(b) *Proof.* Let f be differentiable on a closed interval $[a, b]$ with f' is continuous on $[a, b]$. Further, suppose $|f'(x)| < 1$ on $[a, b]$.

Since f' is continuous on $[a, b]$, it attains both a maximum and a minimum on $[a, b]$. Since $|f'(x)| < 1$ on $[a, b]$, we must have $|f'(x)| \leq M < 1$ on $[a, b]$.

Let $x, y \in [a, b]$. By the Mean Value Theorem, there exists c between x and y so that

$$f(x) - f(y) = f'(c)(x - y),$$

so that

$$|f(x) - f(y)| = |f'(c)||x - y| \leq M|x - y|,$$

with $M < 1$. So, f is a contractive mapping. □

Exercise 5.3.2. Let f be differentiable on an interval A. If $f'(x) \neq 0$ on A, show that f is one-to-one on A. Provide an example to show that the converse statement need not be true.

Solution. *Proof.* Let f be differentiable on an interval A and suppose $f'(x) \neq 0$ on A.

Let $x_1, x_2 \in A$, $x_1 < x_2$. By the Mean Value Theorem, there exists c, $x_1 < c < x_2$, so that

$$\frac{f(x_1) - f(x_2)}{x_1 - x_2} = f'(c).$$

By hypothesis, $f'(c) \neq 0$, so $f(x_1) \neq f(x_2)$. This shows f is one-to-one on A. $\square$

For a nonexample, let $f(x) = x^3$ on the interval $[-1, 1]$. Then f is one-to-one but $f'(0) = 0$.

Exercise 5.3.5. (a) Supply the details for the proof of Cauchy's Generalized Mean Value Theorem (Theorem 5.3.5).

(b) Give a graphical interpretation of the Generalized Mean Value Theorem analogous to the one given for the Mean Value Theorem at the beginning of Section 5.3. (Consider f and g as parametric equations for a curve.)

Solution.

Theorem (Generalized Mean Value Theorem)**.** *If f and g are continuous on the closed interval $[a,b]$ and differentiable on the open interval (a,b), then there exists a point $c \in (a,b)$ where*

$$[f(b) - f(a)]g'(c) = [g(b) - g(a)]f'(c).$$

If g' is never zero on (a,b), then the conclusion can be stated as

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

(a) *Proof.* Let f and g be continuous on the closed interval $[a,b]$ and differentiable on the open interval (a,b).

Define

$$F(x) = f(x)[g(b) - g(a)] - [f(b) - f(a)]g(x).$$

By the hypotheses on f and g, F is continuous on $[a,b]$ and differentiable on the open interval (a,b). By the Mean Value Theorem, there exists c, $a < c < b$, so that

$$0 = F(b) - F(a) = F'(c)(b - a),$$

so $F'(c) = 0$. Computing $F'(c)$, setting it equal to zero, and manipulating gives us

$$[f(b) - f(a)]g'(c) = [g(b) - g(a)]f'(c),$$

as desired.

If g' is never zero on $[a,b]$, then $g(a) \neq g(b)$ by the Mean Value Theorem, and the result can be written

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

□

(b) *Proof.* Consider the parametric curve given by

$$\begin{cases} x = g(t) \\ y = f(t). \end{cases}$$

Then by the chain rule, $dy/dx = f'/g'$. The Generalized Mean Value Theorem says that the slope of the secant line between $(g(a), f(a))$ and $(g(b), f(b))$ equals the slope of the tangent line at some intermediate point $(g(c), f(c))$. □