Homework #10 Solutions
Due Monday, October 20

Exercise 4.4.11. (**Topological Characterization of Continuity**) Let g be defined on all of $\mathbb{R}$. If B is a subset of $\mathbb{R}$, define the set

$$g^{-1}(B) = \{x \in \mathbb{R} : g(x) \in B\}.$$

Show that g is continuous if and only if $g^{-1}(O)$ is open whenever $O \subseteq \mathbb{R}$ is an open set.

Solution. *Proof.* ($\Rightarrow$) Suppose that g is continuous and let $O \subseteq \mathbb{R}$ be an open set.

Let $c \in g^{-1}(O)$. Then $g(c) \in O$. Since O is an open set, we can find $\epsilon > 0$ so that $V_\epsilon(g(c)) \subseteq O$. Since g is continuous at c, we can find $\delta > 0$ so that $|x - c| < \delta$ implies $|g(x) - g(c)| < \epsilon$. This means if $x \in V_\delta(c)$, then $g(x) \in V_\epsilon(g(c))$. This implies that $V_\delta(c) \subseteq g^{-1}(O)$ which shows c is an interior point of $g^{-1}(O)$. Since $c \in g^{-1}(O)$ is arbitrary, $g^{-1}(O)$ is an open set. Since $O \subseteq \mathbb{R}$ is an arbitrary open set, this shows $g^{-1}(O)$ is open whenever $O \subseteq \mathbb{R}$ is an open set.

($\Leftarrow$) Suppose g be defined on all of $\mathbb{R}$ and $g^{-1}(O)$ is open whenever $O \subseteq \mathbb{R}$ is an open set.

Let $c \in \mathbb{R}$ and let $\epsilon > 0$. Then $V_\epsilon(g(c))$ is an open set. By hypothesis, $g^{-1}(V_\epsilon(g(c)))$ is an open set and contains c. It follows there exists $\delta > 0$ so that $x \in V_\delta(c) \subseteq g^{-1}(V_\epsilon(g(c)))$.

Let $|x - c| < \delta$. Then $x \in V_\delta(c) \subseteq g^{-1}(V_\epsilon(g(c)))$, so $g(x) \in (V_\epsilon(g(c)))$. That is, $|g(x) - g(c)| < \epsilon$. Since $\epsilon > 0$ is arbitrary, g is continuous at c, and since c is arbitrary, g is continuous. $\square$

Exercise 4.4.13. (**Continuous Extension Theorem**)

(a) Show that a uniformly continuous function preserves Cauchy sequences; that is, if $f : A \to \mathbb{R}$ is uniformly continuous and $(x_n) \subseteq A$ is a Cauchy sequence, then show $(f(x_n))$ is a Cauchy sequence.

(b) Let g be a continuous function on the open interval (a, b). Prove that g is uniformly continuous on (a, b) if and only if it is possible to define values $g(a)$ and $g(b)$ at the endpoints so that the extended function g is continuous on $[a, b]$. (In the forward direction, first produce candidates for $g(a)$ and $g(b)$, and then show the extended g is continuous.)

Solution. (a) *Proof.* Let $f : A \to \mathbb{R}$ be uniformly continuous and let $(x_n) \subset A$ be a Cauchy sequence. Consider the sequence $(f(x_n))$.

Let $\epsilon > 0$. Since f is uniformly continuous on A, there exists $\delta > 0$ so that $|f(x) - f(y)| < \epsilon$ whenever $|x - y| < \delta$ for any $x, y \in A$. Since (x_n) is a Cauchy sequence, there exists $N \in \mathbb{N}$ so that $|x_n - x_m| < \delta$ whenever $n, m \geq N$.

Let $n, m \geq N$. Then $|x_n - x_m| < \delta$, whereby $|f(x_n) - f(x_m)| < \epsilon$.

Since $\epsilon > 0$, this shows the sequence $(f(x_n))$ is a Cauchy sequence. □

(b) *Proof.* ($\Rightarrow$) Suppose g is continuous on (a, b) and it is possible to define values $g(a)$ and $g(b)$ at the endpoints so that the extended function g is continuous on $[a, b]$.

If the extended function, $\overline{g}$, of g is continuous on $[a, b]$, then $\overline{g}$ is uniformly continuous on $[a, b]$ by Theorem 4.4.7. But then g is uniformly continuous on (a, b) as well.

($\Leftarrow$) Suppose g is uniformly continuous on (a, b). Let (x_n) be a sequence in (a, b) converging to a. Since (x_n) converges, (x_n) is a Cauchy sequence. Since g is uniformly continuous on (a, b), by part (a), the sequence $(g(x_n))$ is also a Cauchy sequence. Hence $(g(x_n))$ converges. Define $g(a) = \lim_{n\to\infty} g(x_n)$.

If (x'_n) is a second sequence in (a, b) converging to a, then $(x_n - x'_n)$ converges to 0. Since g is uniformly continuous on (a, b), we have $(g(x_n) - g(x'_n))$ converges to 0. So, $g(a)$ is well defined.

By the sequential characterization of continuity, Theorem 4.3.2 (iii), g is continuous at a.

Now, do the same for b. □

Exercise 4.5.2. Provide an example of each of the following, or explain why the request is impossible

(a) A continuous function defined on an open interval with range equal to a closed interval.

(b) A continuous function defined on a closed interval with range equal to an open interval.

(c) A continuous function defined on an open interval with range equal to an unbounded closed set different from $\mathbb{R}$.

(d) A continuous function defined on all of $\mathbb{R}$ with range equal to $\mathbb{Q}$.

Solution. (a) Define $f : (-1, 1) \to \mathbb{R}$ by $f(x) = 3$. Then f is continuous and the image is the closed interval $[3, 3]$. (Perhaps that's cheating.)

Here is a better example due to C. Reeves. Consider $f : (-1, \infty) \to \mathbb{R}$ given by $f(x) = |x|$. Then f is defined on an open interval and the range of f is $[0, \infty)$, a closed interval.

(b) This is not possible if the closed interval is bounded, say $[a, b]$, since this set is compact. Since f is continuous, the image of $[a, b]$ will also be compact, hence not open.

So, if we want an example, we have to look at unbounded closed intervals. The entire real line is also a closed interval. In this case, we can use the function $\arctan : \mathbb{R} \to (-\pi, 2, \pi/2)$. This is a continuous function that takes a closed interval to an open interval.

(c) Define $f : (-\pi/2, \pi/2) \to \mathbb{R}$ by $f(x) = \sec x$. The domain is the open interval $(-\pi/2, \pi/2)$ and the image is the unbounded closed set $[1, \infty)$.

(d) This is not possible. The set of real numbers is a connected set and the image of a connected set under a continuous map is connected. However, the set of rational numbers is not connected.

Exercise 4.5.3. A function f is *increasing* on A if $f(x) \leq f(y)$ for all $x < y$ in A. Show that if f is increasing on $[a, b]$ and satisfies the intermediate value property (Definition 4.5.3), then f is continuous on $[a, b]$.

Solution. *Proof.* Let f be increasing on $[a, b]$ and satisfy the intermediate value property. If $f(a) = f(b)$, then f is constant and there is nothing to prove. So, we assume $f(a) < f(b)$.

Let c be a point of (a, b) and let $\epsilon > 0$. (We will deal with the endpoints separately.)

Let (x_n) be a sequence of points in (a, c) converging to c. Since $x_n < c$ for all n, $f(x_n) \leq f(c)$ for all n. Hence, the set $\{f(x_n)\}$ is bounded above. Using the Axiom of Completeness, let L_- be the supremum of this set. We note that $L_- \leq f(c)$ since $f(c)$ is an upper bound for this set.

Suppose $L_- < f(c)$. Then we have $f(x_n) \leq L_- < f(c)$ for all n. Fix one such N. By the intermediate value property, there must exist x with $x_N < x < c$ so that $f(x_N) \leq L_- < f(x) < f(c)$ However, (x_n) converges to c, so for n sufficiently large we must have $x < x_n < c$. But then $L_- < f(x) \leq f(x_n) \leq f(c)$. This contradicts the fact that L_- be the supremum of this set $\{f(x_n)\}$.

This proves that that the sequence $f(x_n)$ converges to $f(c)$. Hence f is continuous at c from the left.

A similar argument shows that f is continuous at c from the right.

For the endpoints, use the first argument for b and the second argument for a.

□

Exercise 4.5.5. (a) Finish the proof of the Intermediate Value Theorem using the Axiom of Completeness started previously.

(b) Finish the proof of the Intermediate Value Theorem using the Nested Interval Property started previously.

Solution. (a) I start by stating the proof as begun in the text.

Proof. To simplify matters a bit, let's consider the special case where f is a continuous function satisfying $f(a) < 0 < f(b)$ and show that $f(c) = 0$ for some $c \in (a, b)$. First let,

$$K = \{x \in [a, b] : f(x) \leq 0\}.$$

Notice that K is bounded above by b, and $a \in K$ so K is not empty. Thus we may appeal to the Axiom of Completeness to assert that $c = \sup K$ exists.

There are three cases to consider:

$$f(c) > 0, \quad f(c) < 0, \quad f(c) = 0.$$

[Here's where our proof starts.]

Suppose $f(c) > 0$. Since f is continuous on $[a, b]$, there exists $\delta > 0$ so that $|f(x) - f(c)| < \frac{1}{2}f(c)$ provided $|x - c| < \delta$. Then, for $x \in (c - \delta, c + \delta)$ we have

$$-\frac{1}{2}f(c) < f(x) - f(c) < \frac{1}{2}f(c)$$

whereby $f(x) > \frac{1}{2}f(c) > 0$. However, by the definition of c as $\sup K$, there exists $x \in K$ with $c - \delta < x < c$, and for this x, $f(x) \leq 0$ by the definition of K. This is a contradiction.

Suppose $f(c) < 0$. Since f is continuous on $[a, b]$, there exists $\delta > 0$ so that $|f(x) - f(c)| < -\frac{1}{2}f(c)$ provided $|x - c| < \delta$. Then, for $x \in (c - \delta, c + \delta)$ we have

$$\frac{1}{2}f(c) < f(x) - f(c) < -\frac{1}{2}f(c)$$

whereby $f(x) < \frac{1}{2}f(c) < 0$. Choose any x with $c < x < c + \delta$. Then $f(x) < 0$, so $x \in K$. This contradicts the fact that c is an upper bound for K.

We conclude that $f(c) = 0$. □

(b) I start by stating the proof as begun in the text.

Proof. Again, consider the special case where $L = 0$ and $f(a) < 0 < f(b)$. Let $I_0 = [a, b]$, and consider the midpoint

$$z = \frac{a+b}{2}.$$

If $f(z) = 0$, we're done. If $f(z) > 0$, then set $a_1 = a$ and $b_1 = z$. If $f(z) < 0$, then set $a_1 = z$ and $b_1 = b$. In either case, the interval $I_1 = [a_1, b_1]$ has the property that f is negative at the left endpoint and positive at the right.

Having constructed $I_n = [a_n, b_n]$ with $f(a_n) < 0$ and $f(b_n) > 0$, let $z = (a_n + b_n)/2$. If $f(z) = 0$, we're done. If $f(z) > 0$, then set $a_{n+1} = a_n$ and $b_{n+1} = z$. If $f(z) < 0$, then set $a_{n+1} = z$ and $b_{n+1} = b_n$. In either case, the interval $I_n = [a_n, b_n]$ has the property that f is negative at the left endpoint and positive at the right.

Notice that the intervals I_n for a nested set of nonempty closed intervals:

$$I_0 \supseteq I_1 \supseteq I_2 \supseteq I_3 \supseteq I_4 \supseteq \cdots$$

By the Nested Interval Property, there exists $c \in \bigcap_{n=1}^{\infty} I_n$.

We note that the diameter of $[a_n, b_n]$ is $|b - a|/2^n$.

Since $a_n \leq c \leq b_n$, we have $0 \leq c - a_n \leq b_n - a_n = \frac{b-a}{2^n}$. By the Squeeze Theorem, (a_n) converges to c. By the sequential property of continuous functions (Theorem 4.3.2), $(f(a_n))$ converges to $f(c)$. Since $f(a_n) < 0$ for all $n \in \mathbb{N}$, we must have $f(c) \leq 0$.

Similarly, since $a_n \leq c \leq b_n$, we have $-\frac{b-a}{2^n} = a_n - b_n \leq c - b_n \leq 0$. By the Squeeze Theorem, (b_n) converges to c. By the sequential property of continuous functions (Theorem 4.3.2), $(f(b_n))$ converges to $f(c)$. Since $f(b_n) > 0$ for all $n \in \mathbb{N}$, we must have $f(c) \geq 0$.

It follows from the two inequalities that $f(c) = 0$, as desired. □