Homework #9
Due Monday, October 13

Exercise 4.3.3. (a) Supply a proof for Theorem 4.3.9 (see below) using the ϵ-δ characterization of continuity.

(b) Give another proof of this theorem using the sequential characterization of continuity (from Theorem 4.3.2 (iii)).

Theorem. *Given $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$, assume that the range $f(A) = \{f(x) \mid x \in A\}$ is contained in the domain B so that the composition $g \circ f(x)) = g(f(x))$ is defined on A. If f is continuous at $c \in A$, and if g is continuous at $f(c) \in B$, then $g \circ f$ is continuous at c.*

Exercise 4.3.9. Assume $h : \mathbb{R} \to \mathbb{R}$ is continuous on $\mathbb{R}$ and let $K = \{x : h(x) = 0\}$. Show that K is a closed set.

Exercise 4.3.11 (Contraction Mapping Theorem). Let f be a function defined on all of $\mathbb{R}$, and assume there is a constant c such that $0 < c < 1$ and

$$|f(x) - f(y) \leq c|x - y|$$

for all x, $y \in \mathbb{R}$.

(a) Show that f is continuous on $\mathbb{R}$.

(b) Pick some point $y_1 \in \mathbb{R}$ and construct the sequence

$$(y_1, f(y_1), f(f(y_1)), \dots).$$

In general, if $y_{n+1} = f(y_n)$, show that the resulting sequence (y_n) is a Cauchy sequence. Hence we may let $y = \lim y_n$.

(c) Prove that y is a fixed point of f (i.e., $f(y) = y$) and that it is unique in this regard.

(d) Finally, prove that if x is *any* arbitrary point in $\mathbb{R}$, then the sequence $(x, f(x), f(f(x)), \dots)$ converges to y defined in (b).

Exercise 4.4.3. Show that $f(x) = 1/x^2$ is uniformly continuous on the set $[1, \infty)$ but not on the set $(0, 1]$.

Exercise 4.4.4. Decide whether each of the following statements is true or false, justifying each conclusion.

(a) If f is continuous on $[a, b]$ with $f(x) > 0$ for all $a \leq x \leq b$, then $1/f$ is bounded on $[a, b]$ (meaning $1/f$ has bounded range).

(b) If f is uniformly continuous on a bounded set A, then $f(A)$ is bounded.

(c) If f is defined on $\mathbb{R}$ and $f(K)$ is compact whenever K is compact, then f is continuous on $\mathbb{R}$.

Exercise 4.4.9 (Lipschitz Functions). A function $f : A \to \mathbb{R}$ is called *Lipschitz* if there exists a bound $M > 0$ such that

$$\left| \frac{f(x) - f(y)}{x - y} \right| \leq M$$

for all x, $y \in A$. Geometrically speaking, a function f is Lipschitz if there is a uniform bound on the magnitude of the slopes of lines drawn through any two points on the graph of f.

(a) Show that if $f : A \to \mathbb{R}$ is Lipschitz, then it is uniformly continuous on A.

(b) Is the converse statement true? Are all uniformly continuous functions necessarily Lipschitz?