

Homework #6

Due Monday, September 22

Exercise 2.6.2. Give an example of each of the following, or argue that such a request is impossible.

- (a) A Cauchy sequence that is not monotone.
- (b) A Cauchy sequence with an unbounded subsequence.
- (c) A divergent monotone sequence with a Cauchy subsequence.
- (d) An unbounded sequence containing a subsequence that is Cauchy.

Exercise 2.6.3. If (x_n) and (y_n) are Cauchy sequences, then one easy way to prove that $(x_n + y_n)$ is Cauchy is to use the Cauchy Criterion. By Theorem 2.6.4, (x_n) and (y_n) must be convergent, and the Algebra Limit Theorem then implies $(x_n + y_n)$ is convergent and hence Cauchy.

- (a) Give a direct argument that $(x_n + y_n)$ is a Cauchy sequence that does not use the Cauchy Criterion or the Algebraic Limit Theorem.
- (b) Do the same for the product $(x_n y_n)$.

Exercise 2.6.7. Exercises 2.4.4 and 2.5.4 establish the equivalence of the Axiom of Completeness and the Monotone Convergence Theorem. They also show the Nested Interval Property is equivalent to these other two in the presence of the Archimedean Property.

- (a) Assume the Bolzano–Weierstrass Theorem is true and use it to construct a proof of the Monotone Convergence Theorem without making any appeal to the Archimedean Property. This shows that BW, AoC, and MCT are all equivalent.
- (b) Use the Cauchy Criterion to prove the Bolzano–Weierstrass Theorem, and find the point in the argument where the Archimedean Property is implicitly required. This establishes the final link in the equivalence of the five characterizations of completeness discussed at the end of Section 2.6.

(c) How do we know it is impossible to prove the Axiom of Completeness starting from the Archimedean Property?

Exercise 2.7.1. Proving the Alternating Series Test (Theorem 2.7.7) amounts to showing that the sequence of partial sums

$$s_n = a_1 - a_2 + a_3 - \cdots \pm a_n$$

converges. (The opening example in Section 2.1 includes a typical illustration of (s_n) .) Different characterizations of completeness lead to different proofs.

- (a) Prove the Alternating Series Test by showing that (s_n) is a Cauchy sequence.
- (b) Supply another proof for this result using the Nested Interval Property (Theorem 1.4.1).
- (c) Consider the subsequences (s_{2n}) and (s_{2n+1}) , and show how the Monotone Convergence Theorem leads to a third proof for the Alternating Series Test.

Exercise 2.7.3. (a) Provide the details for the proof of the Comparison Test (Theorem 2.7.4) using the Cauchy Criterion for Series.

(b) Give another proof for the Comparison Test, this time using the Monotone Convergence Theorem.

Exercise 2.7.4. Give an example of each or explain why the request is impossible referencing the proper theorem(s).

- (a) Two series $\sum x_n$ and $\sum y_n$ that both diverge but where $\sum x_n y_n$ converges.
- (b) A convergent series $\sum x_n$ and a bounded sequence (y_n) such that $\sum x_n y_n$ diverges.
- (c) Two sequences (x_n) and (y_n) where $\sum x_n$ and $\sum (x_n + y_n)$ both converge but $\sum y_n$ diverges.
- (d) A sequence (x_n) satisfying $0 \leq x_n \leq 1/n$ where $\sum (-1)^n x_n$ diverges.