Homework #15
Due Monday, December 1

Exercise 7.2.1. Let f be a bounded function on $[a,b]$, and let P be an arbitrary partition of $[a,b]$. First, explain why $U(f,P) \geq L(f,P)$. Now, prove Lemma 7.2.6.

Lemma. *For any bounded function f on $[a,b]$, it is always the case that $U(f) \geq L(f)$.*

Exercise 7.2.2. Consider $f(x) = 1/x$ over the interval $[1,4]$. Let P be the partition consisting of the points $\{1, 3/2, 2, 4\}$.

(a) Compute $L(f,P)$, $U(f,P)$ and $U(f,P) - L(f,P)$.

(b) What happens to the value of $U(f,P) - L(f,P)$ when we add the point 3 to the partition?

(c) Find a partition P' of $[1,4]$ for which $U(f,P') - L(f,P') < 2/5$.

Exercise 7.2.3 (Sequential Criterion for Integrability)**.** (a) Prove that a bounded function f is integrable on $[a,b]$ if and only if there exists a sequence of partitions $(P_n)_{n=1}^{\infty}$ satisfying

$$\lim_{n\to 0} [U(f,P_n) - L(f,P_n)] = 0.$$

(b) For each n, let P_n be the partition of $[0,1]$ into n equal subintervals. Find formulas for $U(f,P_n)$ and $L(f,P_n)$ if $f(x) = x$. The formula $1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$ will be useful.

(c) Use the sequential criterion for integrability from (a) to show directly that $f(x) = x$ is integrable on $[0,1]$ and compute $\int_0^1 f$.

Exercise 7.2.5. Assume that, for each n, f_n is an integrable function on $[a,b]$. If $(f_n) \to f$ uniformly on $[a,b]$, prove that f is also integrable on this set. (We will see that this conclusion does not necessarily follow if the convergence is pointwise.)

Exercise 7.3.3. Let

$$f(x) = \begin{cases} 1 & \text{if } x = 1/n \text{ for some } n \in \mathbb{N} \\ 0 & \text{otherwise.} \end{cases}$$

Show that f is integrable on $[0,1]$ and compute $\int_0^1 f$.